



# Total restrained domination in claw-free graphs with minimum degree at least two

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## ABSTRACT

Let  $G = (V, E)$  be a graph. A set  $S \subseteq V$  is a total restrained dominating set if every vertex is adjacent to a vertex in  $S$  and every vertex in  $V - S$  is adjacent to a vertex in  $V - S$ . The total restrained domination number of  $G$ , denoted  $\gamma_{tr}(G)$ , is the smallest cardinality of a total restrained dominating set of  $G$ . We will show that if  $G$  is claw-free, connected, has minimum degree at least two and  $G$  is not one of nine exceptional graphs, then  $\gamma_{tr}(G) \leq \frac{4n}{7}$ .

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## 1. Introduction

For notation and graph theory terminology we, in general, follow [6]. Specifically, let  $G = (V, E)$  be a graph with vertex set  $V$  and edge set  $E$ . For a set  $S \subseteq V$ , the subgraph induced by  $S$  is denoted  $\langle S \rangle_G$  or just  $\langle S \rangle$  if the context is clear. If  $G_1$  is an induced subgraph of  $G$ , then  $G - G_1$  will denote the induced subgraph  $\langle V(G) - V(G_1) \rangle$ . If  $\mathcal{K}$  is a set of graphs and  $G$  has a component that is isomorphic to a graph in  $\mathcal{K}$ , then we will say that  $G$  has a component in  $\mathcal{K}$ . The minimum degree (resp., maximum degree) among the vertices of  $G$  is denoted by  $\delta(G)$  (resp.,  $\Delta(G)$ ).

A set  $S \subseteq V$  is a *dominating set* of  $G$ , denoted  $DS$ , if every vertex not in  $S$  is adjacent to a vertex in  $S$ . The *domination number* of  $G$ , denoted  $\gamma(G)$ , is the minimum cardinality of a  $DS$ . The concept of domination in graphs, with its many variations, is now well studied in graph theory. A thorough study of domination appears in [6,7].

A set  $S \subseteq V$  is a *total restrained dominating set*, denoted  $TRDS$ , if every vertex is adjacent to a vertex in  $S$  and every vertex in  $V - S$  is adjacent to a vertex in  $V - S$ . Every graph without isolated vertices has a total restrained dominating set, since  $S = V$  is such a set. The *total restrained domination number* of  $G$ , denoted  $\gamma_{tr}(G)$ , is the minimum cardinality of a  $TRDS$  of  $G$ . If  $|S| = \gamma_{tr}(G)$ , then  $S$  will be referred to as a  $\gamma_{tr}$ -set. Total restrained domination was introduced by Telle and Proskurowski [10], albeit indirectly, as a vertex partitioning problem and further studied, for example, in [1,9,2,3,5,4,8,11]. A specific application of the total restrained domination number of a graph is discussed in [1].

The number of vertices (edges respectively) of a graph is denoted by  $n$  ( $m$  respectively). A graph  $G$  is said to be *claw-free* if for any vertex  $u$  of degree at least three, we have that if  $v, w, x \in N(u)$ , then  $\langle \{v, w, x, u\} \rangle$  is not isomorphic to  $K_{1,3}$ . In the other case,  $\langle \{v, w, x, u\} \rangle$  is a claw and  $u$  is called the center of the claw.

In [8], total restrained domination was studied in graphs with minimum degree at least two. The following bound was derived.

**Theorem 1.** *If  $G$  is a connected graph with  $n \geq 4$ ,  $\delta \geq 2$  and  $\Delta \leq n - 2$  then  $\gamma_{tr}(G) \leq n - \frac{\Delta}{2} - 1$ .*

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In [9] the bound in Theorem 1 was further improved for claw-free graphs with  $\Delta \geq 6$ , with the following result:

**Theorem 2.** *If  $G$  is a connected claw-free graph with  $n \geq 4$ ,  $\delta \geq 2$  and  $\Delta \leq n - 2$  then  $\gamma_{\text{tr}}(G) \leq n - \Delta + 1$ .*

Let  $\mathcal{B}$  be the graph obtained by joining a degree two vertex of a  $C_3$  to the two vertices of a  $K_2$ . Let  $\mathcal{K} = \{\mathcal{B}, C_3, C_5, C_6, C_7, C_{10}, C_{11}, C_{15}, C_{19}\}$ . In [1] the following result is derived:

**Proposition 3.** *If  $n = 4q + r$  for some positive integer  $q$  and  $r \in \{0, 1, 2, 3\}$ , then  $\gamma_{\text{tr}}(C_n) = 2q + r$ .*

**Proposition 4.** *Let  $G$  be a claw-free graph with  $\delta(G) \geq 2$ . The following conditions hold:*

1. *If  $G = \mathcal{B}$  then  $\gamma_{\text{tr}}(G) > \frac{4n}{7}$ .*
2. *If  $G = C_n$  then  $\gamma_{\text{tr}}(C_n) \leq \frac{4n}{7}$  if and only if  $n \notin \{3, 5, 6, 7, 10, 11, 15, 19\}$ .*

Consider a path  $P : x = v_0, v_1, v_2, \dots, v_{j+1} = y$ , where  $\deg(v_i) = 2$ , for  $i = 1, 2, \dots, j$ , and  $\deg(x) \geq 3$  and  $\deg(y) \geq 3$ . The path  $P$  will be called a 2-path. If we set  $x = y$  then  $P$  will be called a 2-cycle. For a 2-path  $P$  define  $\mathcal{H}(P) = \{v \in V(G) - V(P) \mid v \in N(x) \cap N(y)\}$ .

Throughout the paper, if  $G$  has an induced subgraph  $D$  in  $\mathcal{K}$  then we will let the vertices  $u_1, u_2, \dots, u_i$  denote the Hamiltonian path of  $D$  (by default), with  $i \in \{3, 5, 6, 7, 10, 11, 15, 19\}$ , provided that the vertex labels have not already been used. Clearly, if  $D = \mathcal{B}$  then  $\deg(u_3) = 4$ .

Consider a path  $P' : v_1, v_2, \dots, v_j$ , with  $2 \leq j \leq 4$ . Let  $\deg(v_k) = 1$  for  $k \in \{1, j\}$  and  $\deg(v_k) = 2$  for  $k \in \{2, 3, \dots, j-1\}$ . Let  $D \in \mathcal{K}$ . We describe the following construction.

*Construction:*

1. If  $D \in \mathcal{K} - \{\mathcal{B}, C_3\}$  then join  $v_j$  to at least  $u_1$  and  $u_2$  in  $D$ .
2. If  $D = C_3$  and  $j \geq 3$  then join  $v_j$  to at least  $u_1$  in  $D$ .
3. If  $D = \mathcal{B}$  then join  $v_j$  to at least  $u_1$  in  $D$ .

Additional edges may be added between the vertices of  $P' - \{v_j\}$  and  $D$  and between the non-adjacent vertices of  $P'$ .

If a graph is obtained by using the construction mentioned above, we will refer to it as a *necklace*. The path  $P'$  will be referred to as the *attachment*. We will refer to  $v_1$  ( $v_j$  respectively) as the *initial vertex* (*end vertex* respectively) of the necklace.

We are now ready to proceed to the main result of this paper. We shall show the following:

**Theorem 5.** *Let  $G$  be a connected claw-free graph with minimum degree at least two. If  $G \notin \mathcal{K}$ , then  $\gamma_{\text{tr}}(G) \leq \frac{4n}{7}$ .*

## 2. Proof of Theorem 5

We will prove our main result by contradiction. Suppose, to the contrary, that there is at least one connected claw-free graph  $J$  of order  $n$  and minimum degree at least two, such that  $J \notin \mathcal{K}$  and  $\gamma_{\text{tr}}(J) > \frac{4n}{7}$ . Among all the counter examples, choose  $G$  to have minimum size.

The proof of Theorem 5 will follow from a series of key lemmas. We will start by making a few observations that will play an essential part in the proofs of these lemmas.

Suppose that  $G$  has a necklace  $G_1$  as an induced subgraph. For the purpose of the next observation, we will label the vertices of  $G_1$  in exactly the same way as described in our construction. Let  $G_2 = G - G_1$ .

**Observation 6.** *Suppose that  $V(G_2) = \emptyset$ , or  $\delta(G_2) \geq 2$  and  $G_2$  has no components in  $\mathcal{K}$ . Then:*

1. *The attachment of the necklace  $G_1$  has length at most two.*
2. *The initial vertex of the necklace  $G_1$  has no neighbor in  $V(G_2)$ .*

**Proof.** Let  $S$  be any  $\gamma_{\text{tr}}$ -set of  $G_2$ . Note that  $G_2$  is claw-free. Suppose that  $V(G_2) = \emptyset$ , or  $\delta(G_2) \geq 2$  and  $G_2$  has no components in  $\mathcal{K}$ . Suppose, to the contrary, that the attachment either has length three or that it has length at most two, with  $v_1$  having a neighbor  $v$  in  $V(G_2)$ . Note that in the case where the attachment has length three,  $v_1$  has a neighbor  $v$  in  $V(G_2) \cup (V(G_1) - \{v_1, v_2\})$ . Since  $G_2$  has smaller size than  $G$ , it follows that  $|S| \leq \frac{4n(G_2)}{7} = \frac{4(n-i-j)}{7}$ . We will produce a contradiction by showing that  $\gamma_{\text{tr}}(G) \leq \frac{4n}{7}$ . Let  $i = 4q + r$ , where  $q$  is a positive integer and  $r \in \{1, 2, 3\}$ .

*Case 1:  $i = 4q + 3$ .*

We will first look at the case where  $j = 4$ . If  $q \geq 1$  then the set  $S \cup \{v_1, v_2, u_1, u_i\} \cup \bigcup_{k=0}^{q-1} \{u_{4k+3}, u_{4k+4}\}$  is a TRDS of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S| + 4 + 2q \leq \frac{4n}{7} - \frac{4}{7}(4q + 7) + \frac{14q}{7} + \frac{28}{7} \leq \frac{4n}{7}$ . If  $q = 0$  then if  $v \in V(G_2) - S$  or  $v \in \{u_1, v_4\}$  then  $S \cup \{v_2, v_3, u_2, u_3\}$  is a TRDS of  $G$ . If  $v \in S$  or  $v = v_3$  then  $S \cup \{v_3, v_4, u_1\}$  is a TRDS of  $G$ . If  $v \in \{u_2, u_3\}$  then  $S \cup \{v_2, v_3, v_4, u_1\}$  is a TRDS of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S| + 4 \leq \frac{4n}{7} - \frac{28}{7} + \frac{28}{7} = \frac{4n}{7}$ .

Suppose that  $j = 3$ . If  $q \geq 1$  and  $v \notin S$  then the set  $S \cup \{v_2, v_3, u_{i-1}\} \cup \bigcup_{k=0}^{q-1} \{u_{4k+4}, u_{4k+5}\}$  is a TRDS of  $G$ . If  $v \in S$  then the set  $S \cup \{v_3, u_1\} \cup \bigcup_{k=0}^{q-1} \{u_{4k+4}, u_{4k+5}\}$  is a TRDS of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S| + 3 + 2q \leq \frac{4n}{7} - \frac{4}{7}(4q + 6) + \frac{14q}{7} + \frac{21}{7} \leq \frac{4n}{7}$ . If  $q = 0$  then if  $v \in S$  then  $S \cup \{v_3, u_1\}$  is a TRDS of  $G$ . If  $v \notin S$  then  $S \cup \{v_2, v_3, u_1\}$  is a TRDS of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S| + 3 \leq \frac{4n}{7} - \frac{24}{7} + \frac{21}{7} \leq \frac{4n}{7}$ .

Suppose  $j = 2$ . If  $q = 0$  then  $i = 3$ . By the second point of the construction mentioned above, we must have that  $j \geq 3$  which is a contradiction. It follows that  $q \geq 1$ . The set  $S \cup \{v_1, v_2, u_1\} \cup \bigcup_{k=0}^{q-1} \{u_{4k+4}, u_{4k+5}\}$  is a *TRDS* of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S| + 3 + 2q \leq \frac{4n}{7} - \frac{4}{7}(4q + 5) + \frac{14q}{7} + \frac{21}{7} \leq \frac{4n}{7}$ .

Case 2:  $i = 4q + 2$ .

We will first look at the case where  $j = 4$ . If  $v \in V(G_2) - S$  or  $v \in V(G_1) - \{v_1, v_2, v_3, v_4\} - \bigcup_{k=0}^{q-1} \{u_{4k+4}, u_{4k+5}\}$  then the set  $S \cup \{v_2, v_3, v_4\} \cup \bigcup_{k=0}^{q-1} \{u_{4k+4}, u_{4k+5}\}$  is a *TRDS* of  $G$ . If  $v \in S$  or  $v \in \{v_3, v_4\} \cup \bigcup_{k=0}^{q-1} \{u_{4k+4}, u_{4k+5}\}$  then the set  $S \cup \{v_3, v_4\} \cup \bigcup_{k=0}^{q-1} \{u_{4k+4}, u_{4k+5}\}$  is a *TRDS* of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S| + 3 + 2q \leq \frac{4n}{7} - \frac{4}{7}(4q + 6) + \frac{14q}{7} + \frac{21}{7} \leq \frac{4n}{7}$ .

Suppose that  $j = 3$ . If  $v \notin S$  then the set  $S \cup \{v_2, v_3\} \cup \bigcup_{k=0}^{q-1} \{u_{4k+4}, u_{4k+5}\}$  is a *TRDS* of  $G$ . If  $v \in S$  then the set  $S \cup \{v_1, u_1, u_i\} \cup \bigcup_{k=0}^{q-1} \{u_{4k+4}, u_{4k+5}\}$  is a *TRDS* of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S| + 3 + 2q \leq \frac{4n}{7} - \frac{4}{7}(4q + 5) + \frac{14q}{7} + \frac{21}{7} \leq \frac{4n}{7}$ .

Suppose  $j = 2$ . The set  $S \cup \{v_1, v_2\} \cup \bigcup_{k=0}^{q-1} \{u_{4k+4}, u_{4k+5}\}$  is a *TRDS* of  $G$  when  $v \in S$  or  $v \notin S$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S| + 2 + 2q \leq \frac{4n}{7} - \frac{4}{7}(4q + 4) + \frac{14q}{7} + \frac{14}{7} \leq \frac{4n}{7}$ .

Case 3:  $i = 5$ .

We consider first the case where  $D = C_5$ . Suppose first that  $j = 4$ . The set  $S \cup \{v_1, v_2, u_1, u_4, u_5\}$  is a *TRDS* of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S| + 5 \leq \frac{4n}{7} - \frac{36}{7} + \frac{35}{7} \leq \frac{4n}{7}$ .

Suppose that  $j = 3$ . If  $v \notin S$  then the set  $S \cup \{v_2, v_3, u_4, u_5\}$  is a *TRDS* of  $G$ . If  $v \in S$  then the set  $S \cup \{v_3, u_1, u_4, u_5\}$  is a *TRDS* of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S| + 4 \leq \frac{4n}{7} - \frac{32}{7} + \frac{28}{7} \leq \frac{4n}{7}$ .

Suppose  $j = 2$ . If  $v \notin S$  then the set  $S \cup \{u_1, u_5, u_4, v_2\}$  is a *TRDS* of  $G$ . If  $v \in S$  then the set  $S \cup \{u_1, u_5, u_4\}$  is a *TRDS* of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S| + 4 \leq \frac{4n}{7} - \frac{28}{7} + \frac{28}{7} = \frac{4n}{7}$ .

We now consider the case where  $D = \mathcal{B}$ . If  $j = 4$  then the set  $S \cup \{v_1, v_2, u_1, u_2, u_3\}$  is a *TRDS* of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S| + 5 \leq \frac{4n}{7} - \frac{36}{7} + \frac{35}{7} \leq \frac{4n}{7}$ .

If  $j = 3$  then the set  $S \cup \{v_1, v_2, u_2, u_3\}$  is a *TRDS* of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S| + 4 \leq \frac{4n}{7} - \frac{32}{7} + \frac{28}{7} \leq \frac{4n}{7}$ .

Suppose  $j = 2$ . If  $v \notin S$  then the set  $S \cup \{u_1, u_5, u_4, v_2\}$  is a *TRDS* of  $G$ . If  $v \in S$  then the set  $S \cup \{v_1, u_3, u_4, u_5\}$  is a *TRDS* of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S| + 4 \leq \frac{4n}{7} - \frac{28}{7} + \frac{28}{7} = \frac{4n}{7}$ .  $\square$

Consider a path  $P_A : v_1, v_2, v_3, v_4$  of  $G$ , where either  $v_1$  and  $v_4$  are adjacent or  $v_1$  is adjacent to a vertex  $v_0 \in V(G) - V(P_A)$  and  $v_4$  is adjacent to a vertex  $v_5 \in V(G) - V(P_A)$ . Let  $G_2 = G - \langle \{v_1, v_2, v_3, v_4\} \rangle$ . We form the graph  $G_A$  from  $G_2$  as follows. If  $v_1$  is adjacent to  $v_4$  or  $v_0 = v_5$ , then let  $G_A = G_2$ . If  $v_0 \neq v_5$  then either let  $G_A = G_2$  if  $v_0$  is adjacent to  $v_5$  or form  $G_A$  from  $G_2$  by joining  $v_0$  and  $v_5$ . For a path  $P_B : v_1, v_2, v_3, v_4$  where  $v_4$  is adjacent to  $v_2$ , we let  $G_B = G - \langle \{v_1, v_2, v_3, v_4\} \rangle$ .

**Observation 7.** If  $G_A$  ( $G_B$  respectively) is claw-free, has no components in  $\mathcal{K}$  and  $\delta(G_A) \geq 2$  ( $\delta(G_B) \geq 2$  respectively) then  $\gamma_{\text{tr}}(G) \leq \frac{4n}{7}$ , a contradiction.

**Proof.** Suppose that  $G_A$  ( $G_B$  respectively) is claw-free, has no components in  $\mathcal{K}$  and  $\delta(G_A) \geq 2$  ( $\delta(G_B) \geq 2$  respectively). Since  $G_A$  ( $G_B$  respectively) is of smaller size than  $G$  it follows that  $\gamma_{\text{tr}}(G_A) \leq \frac{4(n-4)}{7}$  ( $\gamma_{\text{tr}}(G_B) \leq \frac{4(n-4)}{7}$  respectively).

Now for a  $\gamma_{\text{tr}}$ -set  $S$  of  $G_B$  we have that  $S \cup \{v_1, v_2\}$  is a *TRDS* of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S| + 2 \leq \frac{4n}{7} - \frac{16}{7} + \frac{14}{7} \leq \frac{4n}{7}$ .

For a  $\gamma_{\text{tr}}$ -set  $S$  of  $G_A$  we have that if  $v_0, v_5 \notin S$  or  $v_1$  is adjacent to  $v_4$ , then  $S \cup \{v_2, v_3\}$  is a *TRDS* of  $G$ . If, without loss of generality,  $v_0 \notin S$  and  $v_5 \in S$ , then  $S \cup \{v_1, v_2\}$  is a *TRDS* of  $G$ . If  $v_0, v_5 \in S$  then  $S \cup \{v_1, v_4\}$  is a *TRDS* of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S| + 2 \leq \frac{4n}{7} - \frac{16}{7} + \frac{14}{7} \leq \frac{4n}{7}$ .  $\square$

**Observation 8.** Suppose that  $D \in \mathcal{K}$  is an induced subgraph of  $G$ . If  $v \in V(G) - V(D)$  and  $v$  is adjacent to a vertex of  $D$  then we have the following:

1. If  $D \in \mathcal{K} - \{\mathcal{B}, C_3\}$  then  $v$  is adjacent to two consecutive vertices on the cycle  $D$ .
2. If  $D = \mathcal{B}$  then  $N(v) \cap (V(D) - \{u_3\}) \neq \emptyset$ .

**Proof.** Suppose that  $D$  is an induced subgraph of  $G$ , where  $D \in \mathcal{K}$ . Furthermore, let  $v \in V(G) - V(D)$  where  $v$  is adjacent to a vertex of  $D$ . If  $D \in \mathcal{K} - \{\mathcal{B}, C_3\}$  then suppose that  $v$  is adjacent to say  $u_1$ . Since  $\langle \{v, u_i, u_2, u_1\} \rangle$  is not a claw and  $u_i$  is not adjacent to  $u_2$  we have that  $v$  is adjacent to say  $u_2$ . Hence,  $v$  is adjacent to two consecutive vertices on the cycle  $D$ . If  $D = \mathcal{B}$  and  $v$  is adjacent to  $u_3$  then since  $\langle \{v, u_4, u_1, u_3\} \rangle$  is not a claw and  $u_4$  is not adjacent to  $u_1$  we have that  $v$  is adjacent to a vertex in  $V(D) - \{u_3\}$ .  $\square$

Let  $P : v_1, v_2, v_3, v_4$  be a path of  $G$ . Define  $G_1 = \langle V(P) \rangle$  and  $G_2 = G - G_1$ .

**Observation 9.** Suppose that  $\delta(G_2) \geq 2$  and that  $G_2$  is disconnected. If  $G_2$  consists of exactly two components  $\mathcal{U}$  and  $\mathcal{U}'$  and  $v_1$  ( $v_4$  respectively) is adjacent to a vertex of  $\mathcal{U}$  ( $\mathcal{U}'$  respectively) then  $\mathcal{U} \notin \{C_3, \mathcal{B}\}$  or  $\mathcal{U}' \notin \{C_3, \mathcal{B}\}$ .

**Proof.** Let  $G_2$  be disconnected and let  $\mathcal{U}$  and  $\mathcal{U}'$  be the two components of  $G_2$ . Suppose that  $v_1$  ( $v_4$  respectively) is adjacent to a vertex of  $\mathcal{U}$  ( $\mathcal{U}'$  respectively). Assume, to the contrary, that  $\mathcal{U} \in \{C_3, \mathcal{B}\}$  and  $\mathcal{U}' \in \{C_3, \mathcal{B}\}$ . Let  $P' : u_1, \dots, u_i$  ( $P'' : w_1, \dots, w_j$  respectively) be the Hamiltonian path of  $\mathcal{U}$  ( $\mathcal{U}'$  respectively). Using **Observation 8**, we have that  $v_1$  ( $v_4$  respectively) is adjacent to say  $u_1$  ( $w_1$  respectively).

By symmetry, we need only to consider the case where  $\mathcal{U} = C_3$  and  $\mathcal{U}' \in \{C_3, \mathcal{B}\}$  and the case where  $\mathcal{U} = \mathcal{U}' = \mathcal{B}$ . If  $\mathcal{U} = \mathcal{U}' = C_3$  then  $n(G) = 10$  and the set  $\{v_1, u_1, v_4, w_1\}$  is a TRDS of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq 4 < \frac{4 \cdot 10}{7} = \frac{4n}{7}$  which is a contradiction. If  $\mathcal{U} = C_3$  and  $\mathcal{U}' = \mathcal{B}$  then  $n(G) = 12$  and the set  $\{v_1, u_1, v_4, w_1, w_2, w_3\}$  is a TRDS of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq 6 < \frac{4 \cdot 12}{7} = \frac{4n}{7}$  which is a contradiction. If  $\mathcal{U} = \mathcal{U}' = \mathcal{B}$  then  $n(G) = 14$  and the set  $\{v_1, u_1, u_2, u_3, v_4, w_1, w_2, w_3\}$  is a TRDS of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq 8 < \frac{4 \cdot 14}{7} = \frac{4n}{7}$  which is a contradiction.  $\square$

Suppose that  $D \in \mathcal{K}$  is an induced subgraph of  $G$  and let  $v \in V(G) - V(D)$  be adjacent to a vertex of  $D$ . Define  $G_1 = \langle V(D) \cup \{v\} \rangle$  and  $G_2 = G - G_1$ , and suppose that  $V(G_2) \neq \emptyset$ .

**Observation 10.** *We have the following:*

1. If  $D \in \mathcal{K} - \{\mathcal{B}, C_3\}$  then  $G_2$  has at least one component in  $\mathcal{K}$  or  $\delta(G_2) \leq 1$ .
2. If  $D = \mathcal{B}$  and  $v$  is adjacent to a vertex  $w \in V(G_2)$  then  $G_2$  has at least one component in  $\mathcal{K}$  or  $\delta(G_2) \leq 1$ .

**Proof.** If  $D \in \mathcal{K} - \{\mathcal{B}, C_3\}$  then suppose, to the contrary, that  $G_2$  has no component in  $\mathcal{K}$  and  $\delta(G_2) \geq 2$ . By [Observation 8](#),  $G_1 \notin \mathcal{K}$ . The graph  $G_2$  ( $G_1$  respectively) has a TRDS  $S_2$  ( $S_1$  respectively) of cardinality at most  $\frac{4n(G_2)}{7}$  ( $\frac{4n(G_1)}{7}$  respectively). The set  $S_1 \cup S_2$  is a TRDS of  $G$  and so  $\gamma_{\text{tr}}(G) \leq |S_1| + |S_2| \leq \frac{4n(G_1)}{7} + \frac{4n(G_2)}{7} = \frac{4n}{7}$ , a contradiction. Hence,  $G_2$  has at least one component in  $\mathcal{K}$  or  $\delta(G_2) \leq 1$ .

Suppose that  $D = \mathcal{B}$  and that  $v$  is adjacent to a vertex  $w \in V(G_2)$ . Assume, to the contrary, that  $G_2$  has no component in  $\mathcal{K}$  and  $\delta(G_2) \geq 2$ . The graph  $G_2$  has a TRDS  $S$  of cardinality at most  $\frac{4n(G_2)}{7}$ . By [Observation 8](#),  $v$  is adjacent to say  $u_1$ . If  $w \in S$  ( $w \notin S$  respectively) the set  $S \cup \{u_3, u_2\}$  ( $S \cup \{u_1, u_3, u_2\}$  respectively) is a TRDS of  $G$  and so  $\gamma_{\text{tr}}(G) \leq |S| + 3 \leq \frac{4n}{7} - \frac{24}{7} + \frac{21}{7} < \frac{4n}{7}$ , a contradiction. Hence,  $G_2$  has at least one component in  $\mathcal{K}$  or  $\delta(G_2) \leq 1$ .  $\square$

**Observation 11.** *Let  $e$  be an edge of  $G$ . If the graph  $G - e$  is claw-free and has no component isomorphic to  $C_3$ , then  $\delta(G - e) \leq 1$ .*

**Proof.** Let  $e = xy$  be an edge of  $G$ . Suppose that  $G - e$  is claw-free and that  $G - e$  has no component isomorphic to  $C_3$ . Suppose, to the contrary, that  $\delta(G - e) \geq 2$ .

Case 1:  $G - e$  is connected. If  $G - e \notin \mathcal{K}$  then since  $G - e$  has smaller size than  $G$ , we have that  $G - e$  has a TRDS  $S$  of cardinality at most  $\frac{4n}{7}$ . Since  $S$  is also a TRDS of  $G$  we have that  $\gamma_{\text{tr}}(G) = |S| \leq \frac{4n}{7}$ , a contradiction. Suppose first that  $G - e \in \mathcal{K} - \{\mathcal{B}, C_3\}$ . If  $x$  and  $y$  are at distance more than two apart on  $G - e$ , then  $x, y$  and the two vertices adjacent to  $y$  on  $G - e$  form a claw in  $G$ , which is a contradiction. Thus  $x$  and  $y$  are at distance two apart on  $G - e$ . Hence  $\gamma_{\text{tr}}(G) \leq \frac{4n}{7}$ , which is a contradiction. If  $G - e = \mathcal{B}$  then  $n(G) = 5$  and  $\gamma_{\text{tr}}(G) = 2 < \frac{4n}{7}$ , a contradiction.

Case 2:  $G - e$  is disconnected. Let  $D$  and  $D'$  be the two components of  $G - e$ . Neither  $D$  nor  $D'$  is in  $\mathcal{K} - \{\mathcal{B}, C_3\}$ , since otherwise  $G$  will not be claw-free. Let  $x \in V(D)$  and  $y \in V(D')$ . Certainly,  $D$  is not isomorphic to  $C_3$ . If  $D = \mathcal{B}$  then, by [Observation 8](#),  $x$  is say  $u_1$ . Note that  $G' = G - xu_3$  is claw-free, connected,  $G' \notin \mathcal{K}$  and  $\delta(G') \geq 2$ . Furthermore,  $G'$  has smaller size than  $G$  and so  $G'$  has a TRDS  $S$  of cardinality at most  $\frac{4n(G')}{7}$ . Since  $S$  is also a TRDS of  $G$  we have that  $\gamma_{\text{tr}}(G) \leq \frac{4n}{7}$ , which is a contradiction. Hence, neither  $D$  nor  $D'$  is in  $\mathcal{K}$ . Since  $D$  ( $D'$  respectively) has smaller size than  $G$  it follows that  $D$  ( $D'$  respectively) has a TRDS  $S_1$  ( $S_2$  respectively) of cardinality at most  $\frac{4n(D)}{7}$  ( $\frac{4n(D')}{7}$  respectively). Then  $S_1 \cup S_2$  is a TRDS of  $G$ , and so  $\gamma_{\text{tr}}(G) \leq |S_1| + |S_2| \leq \frac{4n(D)}{7} + \frac{4n(D')}{7} = \frac{4n}{7}$  which is a contradiction.  $\square$

To complete the proof of [Theorem 5](#) we will first state our key lemmas, then briefly explain the importance of each lemma and then use them to prove our main result. The proof of each of the following lemmas will be provided in the next section and each proof will rely on the observations mentioned above.

**Lemma 12.** *The graph  $G$  has no 2-paths of length greater than three.*

**Lemma 13.** *Suppose that  $G$  has a path  $P : v_1, v_2, v_3, v_4$  of which  $\deg(v_1) = \deg(v_4) = 2$ . If  $G_1 = \langle V(P) \rangle$  and  $G_2 = G - G_1$  then  $\delta(G_2) \leq 1$ .*

**Lemma 14.** *Suppose that  $G$  has a path  $P : v_1, v_2, v_3, v_4$  of which the vertices  $v_1$  and  $v_4$  have degree at least two and  $\deg(v_3) = 2$ . If  $G_1 = \langle \{v_1, v_2, v_3, v_4\} \rangle$  and  $G_2 = G - G_1$  then  $\delta(G_2) \leq 1$ .*

**Lemma 15.** *The graph  $G$  has no 2-paths of length three.*

**Lemma 16.** *The graph  $G$  either has no 2-cycles or it has a 2-path of length two.*

**Lemma 17.** *If  $P$  is a 2-path of  $G$  that has length two then  $\mathcal{H}(P) = \emptyset$ .*

**Lemma 18.** *The graph  $G$  has no 2-paths of length two.*

**Lemma 19.** *The graph  $G$  does not have minimum degree at least three.*

The proof of [Theorem 5](#) will follow from [Lemmas 16, 18 and 19](#). [Lemmas 12–15](#) will be used to prove [Lemmas 16–18](#). [Lemma 17](#) will be essential in the proof of [Lemma 18](#). We are now ready to complete the proof of [Theorem 5](#).

Suppose first that  $\delta(G) = 2$ . If  $G = C_n$  then [Proposition 4](#) provides a contradiction. Hence,  $G$  has either a 2-path or a 2-cycle. If  $G$  has a 2-cycle then, by [Lemma 16](#), we have that  $G$  must have a 2-path of length two. But this fact contradicts [Lemma 18](#). Hence,  $G$  has no 2-cycles. It follows, by [Lemmas 12 and 15](#), that  $G$  must have a 2-path of length two and so we contradict [Lemma 18](#). Hence,  $\delta(G) \geq 3$  and so we contradict [Lemma 19](#). It immediately follows that  $G$  does not exist and so we obtain a contradiction which proves [Theorem 5](#).  $\square$

### 3. Proofs of key lemmas

**Proof of Lemma 12.** Suppose, to the contrary, that  $G$  has a 2-path  $P : x = v_0, v_1, v_2, \dots, v_{j+1} = y$ , with  $j \geq 3$ .

Suppose that  $j \geq 4$ .

If  $j \geq 5$  or  $j = 4$  and  $x$  is not adjacent to  $y$  then we let  $G_2 = G - \langle \{v_1, v_2, \dots, v_4\} \rangle$  and we form  $G_A$  from  $G_2$  by joining the vertices  $x$  and  $v_5$ . The graph  $G_A$  is claw-free and  $\delta(G_A) \geq 2$ . Furthermore,  $G_A$  is connected and  $G_A \notin \mathcal{K}$ . By [Observation 7](#) we get a contradiction.

Hence,  $j = 4$  and  $x$  is adjacent to  $y$ . Let  $G_2 = G - \langle \{v_1, v_2, \dots, v_4\} \rangle$ . The graph  $G_2$  is claw-free, connected and  $\delta(G_2) \geq 2$ . If  $G_2 \notin \mathcal{K}$  then, by [Observation 7](#), we get a contradiction. Hence,  $G_2 \in \mathcal{K}$ .

If  $G_2 \in \{C_3, \mathcal{B}\}$  then  $G$  is a necklace with an attachment that has length three. By the first part of [Observation 6](#) we get a contradiction. Hence,  $G_2 \in \mathcal{K} - \{C_3, \mathcal{B}\}$ . For  $u \in N(x) - \{v_1, y\}$  the graph  $\langle \{u, v_1, y, x\} \rangle$  is a claw, which is a contradiction.

Hence, we may assume that  $j = 3$ .

**Claim 1.** The only degree two vertex adjacent to  $x$  ( $y$  respectively) is  $v_1$  ( $v_3$  respectively).

**Proof.** Let  $u \in N(y) \cap (V(G) - V(P))$  and suppose, to the contrary, that  $\deg(u) = 2$ .

**Case 1.** The vertex  $u$  is adjacent to  $x$ .

If  $x$  ( $y$  respectively) has a neighbor  $v \in V(G) - V(P) - \{u\}$  ( $v' \in V(G) - V(P) - \{u\}$  respectively), then the graph  $\langle \{u, v, v_1, x\} \rangle$  ( $\langle \{u, v', v_3, y\} \rangle$  respectively) induces a claw, a contradiction. Hence,  $N(x) = \{u, v_1, y\}$  and  $N(y) = \{u, v_3, x\}$ . Furthermore,  $n(G) = 6$  and the set  $\{x, v_1, v_2\}$  is a *TRDS* of  $G$ . Hence,  $\gamma_{tr}(G) \leq 3 < \frac{4.6}{7} = \frac{4n}{7}$  which is a contradiction.

**Case 2.** The vertex  $u$  is not adjacent to  $x$ .

If  $x$  is adjacent to  $y$  then  $\langle \{u, x, v_3, y\} \rangle$  is a claw which is a contradiction. Thus,  $x$  and  $y$  are not adjacent. The fact that  $N(y) - \{v_3\}$  is complete, implies that  $\deg(y) = 3$ . Let  $u' \in N(y) - \{v_3, u\}$ . Clearly,  $u'$  is adjacent to  $u$ . The graph  $G' = G - yu'$  is claw-free and connected. If  $\deg(u') \geq 3$  then  $\delta(G') \geq 2$  and this contradicts [Observation 11](#). Hence,  $\deg(u') = 2$ . Note that the graph  $G_1 = \langle V(P) \cup \{u, u'\} - \{x\} \rangle$  is a necklace, where the path  $P - x - y$  is the attachment with initial vertex  $v_1$ . Let  $G_2 = G - G_1$ . Now  $G_2$  is connected and  $\delta(G_2) \geq 2$ . Note that  $v_1$  has a neighbor in  $V(G_2)$ . If  $G_2 \notin \mathcal{K}$  then we contradict [Observation 6](#). Hence,  $G_2 \in \mathcal{K}$ . Since  $\deg(v_1) = 2$  we have, by the first part of [Observation 8](#), that  $G_2 \in \{C_3, \mathcal{B}\}$ .

If  $G_2 = C_3$  then  $n(G) = 9$  and the set  $\{x, v_1, v_2, v_3, y\}$  is a *TRDS* of  $G$  and so  $\gamma_{tr}(G) \leq 5 < \frac{4.9}{7} = \frac{4n}{7}$ , a contradiction.

If  $G_2 = \mathcal{B}$  we have, by [Observation 8](#), that  $x$  is say  $u_1$ . The graph  $G' = G - xu_3$  is claw-free and connected. In addition,  $\delta(G') \geq 2$  and this contradicts [Observation 11](#). Our claim follows by symmetry.  $\square$

Let  $G_1 = \langle V(P) - \{x\} \rangle$  and let  $G_2 = G - G_1$ . Note that  $\delta(G_2) \geq 1$ .

**Case 1.**  $\delta(G_2) = 1$ .

Let  $v$  be a degree one vertex of  $G_2$ . If  $v \in V(G) - V(P)$  then  $v$  is adjacent to  $y$  and  $\deg(v) = 2$ , contradicting [Claim 1](#). It follows that  $v = x$ . Hence, for some  $u \in N(x) \cap (V(G) - V(P))$  we have that  $N(x) = \{y, v_1, u\}$ . Now if  $\mathcal{H}(P) = \emptyset$  then  $G' = G - xy$  is claw-free, connected and has minimum degree at least two. This contradicts [Observation 11](#). Hence,  $u \in \mathcal{H}(P)$  and  $\deg(u) \geq 3$ . If  $y$  has a neighbor  $u' \in N(y) - \{v_3, x, u\}$  then  $\langle \{x, v_3, u', y\} \rangle$  induces a claw, which is a contradiction. Hence,  $N(y) = \{x, v_3, u\}$ .

**Case 1.1.**  $\deg(u) = 3$ .

Let  $w_1 \in N(u) - \{y, x\}$ . In this case  $N(u) = \{w_1, x, y\}$ . Assume first that  $\deg(w_1) \geq 3$ . Let  $G'_1 = \langle V(P) \cup \{u\} \rangle$  and  $G'_2 = G - G'_1$ . Note that  $G'_2$  is claw-free, connected and  $\delta(G'_2) \geq 2$ . Suppose first that  $G'_2 \in \mathcal{K}$ . By [Observation 8](#) and the fact that  $\deg(u) = 3$ , we have that  $G'_2 \in \{C_3, \mathcal{B}\}$ . [Observation 8](#) implies that say  $w_1 = u_1$ .

If  $G'_2 = C_3$  then  $n(G) = 9$  and  $\{u_1, u, x, v_1, v_2\}$  is a *TRDS* of  $G$  and so  $\gamma_{tr}(G) \leq 5 < \frac{4.9}{7} = \frac{4n}{7}$ , a contradiction.

If  $G'_2 = \mathcal{B}$  then the graph  $G' = G - u_1u_3$  is claw-free, connected and has minimum degree at least two. This contradicts [Observation 11](#).

Hence,  $G'_2 \notin \mathcal{K}$ . Note that  $G'_2$  has smaller size than  $G$  and so  $G'_2$  has a *TRDS*  $S_2$  with cardinality at most  $\frac{4(n-6)}{7}$ . The set  $S_2 \cup \{x, v_1, v_2\}$  is a *TRDS* of  $G$ . Hence,  $\gamma_{tr}(G) \leq |S_2| + 3 \leq \frac{4n}{7} - \frac{24}{7} + \frac{21}{7} < \frac{4n}{7}$  which is a contradiction.

We may assume that  $\deg(w_1) = 2$ . Let  $w_2 \in N(w_1) - \{u\}$ . Since  $N(u) = \{x, y, w_1\}$  we can find a 2-path  $P' : u, w_1, w_2, \dots, w_{j+1}$ , where  $w_i \in V(G) - V(P) - \{u\}$  for  $i = 2, \dots, j+1$ . Since every 2-path has length at most four we have that  $j = 1, 2, 3$ . We let  $G'_1 = \langle V(P) \cup V(P') - \{w_{j+1}\} \rangle$  and  $G'_2 = G - G'_1$ . Note that  $G'_2$  is claw-free and  $\delta(G'_2) \geq 2$ . Also,  $w_j$  has a neighbor in  $V(G'_2)$ . Suppose first that  $G'_2 \notin \mathcal{K}$ . If  $j \geq 2$  then  $G'_1$  is a necklace and so we contradict [Observation 6](#). Hence,  $j = 1$ . The graph  $G'_2$  has a *TRDS*  $S_2$  of cardinality at most  $\frac{4(n-7)}{7}$ . If  $w_2 \in S_2$  then  $\{x, v_1, v_2\} \cup S_2$  is a *TRDS* of  $G$ . If  $w_2 \notin S_2$



then  $\{x, v_1, v_2, u\} \cup S_2$  is a *TRDS* of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S_2| + 4 \leq \frac{4n}{7} - \frac{28}{7} + \frac{28}{7} = \frac{4n}{7}$  which is a contradiction. Therefore,  $G'_2 \in \mathcal{K}$ .

Since  $\deg(w_j) = 2$  we have, by [Observation 8](#), that  $G'_2 \in \{C_3, \mathcal{B}\}$ . Let  $G''_1 = \langle V(G'_2) \cup V(P') - \{u\} \rangle$  and  $G''_2 = G - G'_1$ . Note that if  $G'_2 = \mathcal{B}$  and  $j \geq 2$  or if  $G'_2 = C_3$  and  $j = 3$  then the graph  $G''_1$  is a necklace with attachment  $P' - u - w_{j+1}$  and initial vertex  $w_1$ . If  $j = 1$  and  $G'_2 = C_3$  then by [Observation 7](#) we get a contradiction. If  $j = 1$  and  $G'_2 = \mathcal{B}$  then by [Observation 8](#) we have that  $w_2$  is say  $u_1$ . Furthermore,  $G' = G - u_1 u_3$  is claw-free, connected and  $\delta(G') \geq 2$ . This contradicts [Observation 11](#). If  $j = 3$  or  $j = 2$  and  $G'_2 = \mathcal{B}$ , then we contradict [Observation 6](#). Hence,  $j = 2$  and  $G'_2 = C_3$ . Note that  $n(G) = 11$  and so  $\{w_2, w_3, y, v_3, v_2\}$  is a *TRDS* of  $G$  and so  $\gamma_{\text{tr}}(G) \leq 5 < \frac{4 \cdot 11}{7} = \frac{4n}{7}$ , a contradiction.

Case 1.2.  $\deg(u) \geq 4$ .

Let  $w, w' \in N(u) - \{x, y\}$ . Since  $G$  is claw-free we have that  $\langle N(u) - \{x, y\} \rangle$  is complete. If  $\deg(w) = 2$  then  $w$  is adjacent to  $w'$  and  $\deg(u) = 4$ . If  $\deg(w') \geq 3$  then  $G' = G - uw'$  is claw-free, connected and  $\delta(G') \geq 2$ . This contradicts [Observation 11](#). Hence,  $\deg(w') = 2$  and so  $n(G) = 8$ . The set  $\{u, x, v_1, v_2\}$  is a *TRDS* of  $G$  and so  $\gamma_{\text{tr}}(G) \leq 4 < \frac{4 \cdot 8}{7} = \frac{4n}{7}$ , a contradiction. Hence,  $u$  is not adjacent to any degree two vertex and so if we define  $G'_1 = \langle V(P) \cup \{u\} \rangle$  and  $G'_2 = G - G'_1$  we see that  $\delta(G'_2) \geq 2$ . Now since  $\langle N(u) - \{x, y\} \rangle$  is complete we have that  $G'_2$  is connected. If  $G'_2 \notin \mathcal{K}$  then note that  $G'_2$  has smaller size than  $G$  and so  $G'_2$  has a *TRDS*  $S_2$  of cardinality at most  $\frac{4(n-6)}{7}$ . The set  $S_2 \cup \{x, v_1, v_2\}$  is a *TRDS* of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S_2| + 3 \leq \frac{4n}{7} - \frac{24}{7} + \frac{21}{7} < \frac{4n}{7}$  which is a contradiction. It follows that  $G'_2 \in \mathcal{K}$ . By [Observation 8](#),  $u$  is adjacent to say  $u_1$ .

If  $G'_2 \in \mathcal{K} - \{C_3, \mathcal{B}\}$  then, by [Observation 8](#),  $u$  is adjacent to say  $u_2$ . If there are edges between the vertex  $u$  and the set  $V(G'_2) - \{u_1, u_2\}$ , then we may form  $G'$ , from  $G$ , by removing all these edges. The graph  $G'$  is claw-free, connected,  $G' \notin \mathcal{K}$  and  $G'$  has smaller size than  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq \frac{4n}{7}$  which is a contradiction. It can be concluded that  $P : u_2, u_3, \dots, u_i, u_1$  is a 2-path. By all previous arguments we have that  $i = 5$ . Hence,  $n(G) = 11$  and so  $\{y, v_3, v_2, u_1, u_2, u_3\}$  is a *TRDS* of  $G$ . Thus,  $\gamma_{\text{tr}}(G) \leq 6 < \frac{4 \cdot 11}{7} = \frac{4n}{7}$  which is a contradiction.

If  $G'_2 = C_3$  then  $n(G) = 9$  and so  $\{y, v_3, v_2, u_1, u\}$  is a *TRDS* of  $G$ , and so  $\gamma_{\text{tr}}(G) \leq 5 < \frac{4 \cdot 9}{7} = \frac{4n}{7}$  which is a contradiction.

If  $G'_2 = \mathcal{B}$  then clearly  $n(G) = 11$ . Constructing a  $\gamma_{\text{tr}}$ -set of  $G$  of cardinality at most six will suffice since then  $\gamma_{\text{tr}}(G) \leq 6 < \frac{4 \cdot 11}{7} = \frac{4n}{7}$ , a contradiction. The set  $\{y, v_3, v_2, u_1, u_2, u_3\}$  is a *TRDS* of  $G$ .

Case 2.  $\delta(G_2) \geq 2$ .

If  $G_2$  has no components in  $\mathcal{K}$  then, by [Observation 7](#), we get a contradiction. We may assume that  $G_2$  has a component  $\mathcal{U} \in \mathcal{K}$ .

Case 2.1. The vertex  $x$  lies on the component  $\mathcal{U}$ .

Note that since  $\delta(G_2) \geq 2$  we have that  $x$  is adjacent to a vertex  $u \in V(\mathcal{U})$ . If  $x$  is adjacent to a vertex  $w \in V(G) - V(\mathcal{U}) - \{y, v_1, v_2, v_3\}$  then  $\langle \{w, u, v_1, x\} \rangle$  is a claw which is a contradiction. Hence,  $N[x] \subseteq V(\mathcal{U}) \cup \{v_1, y\}$ . Since  $\deg(v_1) = 2$  we have, by [Observation 8](#), that  $\mathcal{U} \in \{C_3, \mathcal{B}\}$ .

By [Observation 8](#),  $x$  is say  $u_1$ . Assume first that  $y$  is adjacent to a vertex  $z \in V(\mathcal{U})$ . Note that  $N[z] \subseteq V(\mathcal{U}) \cup \{v_1, y\}$ . If  $y$  has a neighbor  $w \in V(G) - V(P) - V(\mathcal{U})$  then  $\langle \{v_3, z, w, y\} \rangle$  would induce a claw, a contradiction. Hence,  $N(y) \subseteq V(\mathcal{U}) \cup \{v_3\}$ . It follows that  $G$  is a necklace with attachment  $P - x$  and initial vertex  $y$ . Since the attachment has length three, we contradict [Observation 6](#). Thus,  $y$  is adjacent to no vertex of  $V(\mathcal{U})$  and so  $u_2$  will have degree two in  $G$ . This contradicts Claim 1.

Case 2.2. The vertex  $x$  does not lie on the component  $\mathcal{U}$ .

By the previous case we may assume that  $x$  lies on a component that is not in  $\mathcal{K}$ . Furthermore,  $y$  is adjacent to some vertex of  $\mathcal{U}$  and  $V(\mathcal{U}) \subseteq V(G) - V(P)$ . Using [Observation 8](#), we can assume that  $y$  is adjacent to say  $u_1$ . If  $\mathcal{U} \in \mathcal{K} - \{C_3, \mathcal{B}\}$  then  $y$  must be adjacent to  $u_2$ . Also note that if  $y$  has a neighbor  $u \in V(G_2) - V(\mathcal{U})$  then  $\langle \{u, u_1, v_3, y\} \rangle$  induces a claw which is a contradiction. Hence,  $N(y) \subseteq \{v_3\} \cup V(\mathcal{U})$ . The graph  $G'_1 = \langle V(\mathcal{U}) \cup V(P) - \{x\} \rangle$  is a necklace with attachment  $P - x$  and initial vertex  $v_1$ . The graph  $G'_2 = G - G'_1$  is claw-free, has no components in  $\mathcal{K}$ ,  $v_1$  has a neighbor in  $V(G'_2)$  and  $\delta(G'_2) \geq 2$ . By [Observation 6](#) we get a contradiction.  $\square$

**Proof of Lemma 13.** Let  $P : v_1, v_2, v_3, v_4$  be a path of which  $\deg(v_1) = \deg(v_4) = 2$ . Define  $G_1 = \langle V(P) \rangle$  and  $G_2 = G - G_1$ . If  $V(G_2) = \emptyset$  then  $\gamma_{\text{tr}}(G) \leq \frac{4n}{7}$ , a contradiction. We may assume, to the contrary, that  $\delta(G_2) \geq 2$ . If  $G_2$  has no component in  $\mathcal{K}$  then, by [Observation 7](#), we have that  $G$  has a *TRDS* of cardinality at most  $\frac{4n}{7}$ , a contradiction. Hence,  $G_2$  has a component  $\mathcal{U}$  in  $\mathcal{K}$ . The following claim will be useful.

**Claim 1.** Let  $D$  be a component of  $G_2$  that is in  $\mathcal{K}$ . If  $D$  has a vertex adjacent to  $v_2$ , neither  $v_1$  nor  $v_4$  is adjacent to a vertex of  $D$  and  $v_1$  is not adjacent to  $v_3$ , then  $D \in \{C_3, \mathcal{B}\}$ .

**Proof.** Assume that  $v_2$  is adjacent to a vertex of  $D$ , neither  $v_1$  nor  $v_4$  is adjacent to a vertex of  $D$  and that  $v_1$  and  $v_3$  are not adjacent. Suppose, to the contrary, that  $D \in \mathcal{K} - \{C_3, \mathcal{B}\}$ . By [Observation 8](#),  $v_2$  must be adjacent to say  $u_1$  and  $u_2$ . In addition, the fact that  $\langle \{v_3, u_1, v_1, v_2\} \rangle$  ( $\langle \{v_3, u_2, v_1, v_2\} \rangle$  respectively) is not a claw, implies that  $v_3$  is adjacent to both  $u_1$  and  $u_2$ . Furthermore, if  $v_2$  is adjacent to  $u_j$  where  $j = 3, \dots, i$ , then either  $\langle \{v_1, u_j, u_1, v_2\} \rangle$  or  $\langle \{v_1, u_j, u_2, v_2\} \rangle$  will induce a claw which is a contradiction. By symmetry we have that  $v_3$  is not adjacent to  $u_j$ , with  $j = 3, \dots, i$ . Hence, the path  $P' : u_2, u_3, \dots, u_i, u_1$  is a 2-path of length greater than three and so we contradict [Lemma 12](#). Hence,  $D \in \{C_3, \mathcal{B}\}$ .  $\square$

Case 1. The vertex  $v_1$  is adjacent to a vertex of  $\mathcal{U}$ .

Since  $\deg(v_1) = 2$ , we have, by [Observation 8](#), that  $\mathcal{U} \in \{C_3, \mathcal{B}\}$  and  $v_1$  is adjacent to say  $u_1$ .

Case 1.1.  $V(G_2) - V(\mathcal{U}) = \emptyset$ .

Note that  $G_2 = \mathcal{U}$ . Hence,  $G$  is a necklace with an attachment that has length three. By the first part of [Observation 6](#) we get a contradiction.

Case 1.2.  $V(G_2) - V(\mathcal{U}) \neq \emptyset$ .

Let  $z \in V(G_2) - V(\mathcal{U})$ .

Case 1.2.1.  $v_2$  is adjacent to either  $z$  or  $v_4$ .

Let  $G'_1 = \langle \{v_1\} \cup V(\mathcal{U}) \rangle$  and  $G'_2 = G - G'_1$ . Clearly,  $G'_2$  is connected and  $\delta(G'_2) \geq 1$ . Note that if  $v_2$  is adjacent to  $z$  then since  $\langle \{z, v_1, v_3, v_2\} \rangle$  does not induce a claw, we have that  $v_3$  is adjacent to  $z$  and so  $G'_2$  has a triangle. If  $v_2$  is adjacent to  $v_4$  then  $G'_2$  contains a triangle. We may therefore assume that  $G'_2$  has an induced triangle.

Suppose first that  $\delta(G'_2) \geq 2$ . If  $G'_2 \in \mathcal{K}$  then since  $G'_2$  has a triangle and at least four vertices, we must have that  $G'_2 = \mathcal{B}$ . Now  $G'_2$  has a TRDS  $S$  of cardinality three. If  $\mathcal{U} = C_3$ , then  $n(G) = 9$ . The set  $\{v_1, u_1\} \cup S$  is a TRDS of  $G$  and so  $\gamma_{\text{tr}}(G) \leq 5 < \frac{4.9}{7} = \frac{4n}{7}$ , which is a contradiction. If  $\mathcal{U} = \mathcal{B}$  then  $n(G) = 11$ . If  $v_2 \in S$  ( $v_2 \notin S$  respectively) we have that  $\{u_3, u_2\} \cup S$  ( $\{u_3, u_2, u_1\} \cup S$  respectively) is a TRDS of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq 6 < \frac{4.11}{7} = \frac{4n}{7}$  which is a contradiction.

We may assume that  $G'_2 \notin \mathcal{K}$ . If  $\mathcal{U} = \mathcal{B}$  then since  $v_1$  is adjacent to the vertex  $v_2 \in V(G'_2)$ , we have that the second part of [Observation 10](#) implies that  $\delta(G'_2) \leq 1$  or  $G'_2 \in \mathcal{K}$  which is a contradiction. It follows that  $\mathcal{U} = C_3$ . Since  $G'_1$  has a Hamiltonian path and  $n(G'_1) = 4$  we obtain, by [Observation 7](#), a contradiction.

We may assume that  $\delta(G'_2) = 1$ .

Since  $\deg_{G'_2}(v_2) \geq 2$  and  $\deg_{G'_2}(v_3) \geq 2$ , we have that  $\deg_{G'_2}(v_4) = 1$ . Hence,  $v_4$  must be adjacent to a vertex of  $\mathcal{U}$  and so  $v_2$  is adjacent to  $z$ . Recall that  $v_3$  is also adjacent to  $z$ . If  $v_4$  is adjacent to  $u_1$  then  $\langle \{v_4, v_1, u_3, u_1\} \rangle$  induces a claw. Hence, if  $\mathcal{U} = C_3$  then  $v_4$  must be adjacent to say  $u_2$ . If  $\mathcal{U} = \mathcal{B}$  then since  $\deg(v_4) = 2$ , we have, by [Observation 8](#), that  $v_4$  is adjacent to  $u_2$  or say  $u_4$ .

Let  $G''_1 = \langle V(P) \cup V(\mathcal{U}) \rangle$  and  $G''_2 = G - G''_1$ .

Suppose first that  $G''_2$  is disconnected. Let  $\mathcal{U}'$  ( $\mathcal{U}''$  respectively) be a component of  $G''_2$  that contains (does not contain respectively) the vertex  $z$ . Without loss of generality,  $v_2$  is adjacent to some  $z' \in V(\mathcal{U}')$  and so  $\langle \{z', z, v_1, v_2\} \rangle$  induces a claw which is a contradiction. Hence,  $G''_2$  is connected. Since  $\delta(G_2) \geq 2$ , we must have that  $\delta(G''_2) \geq 2$ . Note that  $\delta(G'_1) \geq 2$ .

Furthermore, the fact that  $G'_1 \notin \mathcal{K}$  implies that  $G'_1$  has a TRDS  $S_1$  of cardinality at most  $\frac{4n(G'_1)}{7}$ .

If  $G''_2 \notin \mathcal{K}$  then  $G''_2$  has a TRDS  $S_2$  of cardinality at most  $\frac{4n(G''_2)}{7}$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S_1| + |S_2| \leq \frac{4n(G'_1)}{7} + \frac{4n(G''_2)}{7} = \frac{4n}{7}$  which is a contradiction.

Hence,  $G''_2 \in \mathcal{K}$ . Let  $P' : w_1, \dots, w_j$  be the Hamiltonian path of  $G''_2$ . Now  $G''_2$  is clearly a component of  $G_2$ . Furthermore, every vertex of  $G''_2$  is adjacent to neither  $v_1$  nor  $v_4$ ,  $v_1$  is not adjacent to  $v_3$  and  $v_2$  is adjacent to the vertex  $z$  in  $V(G''_2)$ . By applying Claim 1 to the component  $G''_2$ , we have that  $G''_2 \in \{C_3, \mathcal{B}\}$ .

Suppose that  $G''_2 = \mathcal{B}$ . Note that  $G''_2$  has a TRDS  $S$  of cardinality three. If  $\mathcal{U} = \mathcal{B}$  and  $v_4$  is adjacent to  $u_2$  then  $\{u_3, u_4, u_5, v_2, v_3\} \cup S$  is a TRDS of  $G$ . If  $\mathcal{U} = \mathcal{B}$  and  $v_4$  is adjacent to  $u_4$  then  $\{u_3, u_5, v_2, v_3\} \cup S$  is a TRDS of  $G$ . Clearly,  $n(G) = 14$  and  $\gamma_{\text{tr}}(G) \leq 5 + |S| = 8 = \frac{4.14}{7} = \frac{4n}{7}$  which is a contradiction. If  $\mathcal{U} = C_3$  then  $n(G) = 12$ . By [Observation 8](#), we may assume that  $v_2$  is adjacent to say  $w_1$ . The fact that  $\langle \{w_1, v_1, v_3, v_2\} \rangle$  is not a claw implies that  $v_3$  is adjacent to  $w_1$  also. The set  $\{w_1, w_2, w_3, u_1, u_2, u_3\}$  is a TRDS of  $G$  and so  $\gamma_{\text{tr}}(G) \leq 6 < \frac{4.12}{7} = \frac{4n}{7}$ , which is a contradiction.

Suppose that  $G''_2 = C_3$ . If  $\mathcal{U} = \mathcal{B}$  then  $n(G) = 12$ . Without loss of generality, let  $z = w_1$ . If  $v_4$  is adjacent to  $u_2$  then  $\{z, v_2, v_3, u_3, u_4, u_5\}$  is a TRDS of  $G$ . If  $v_4$  is adjacent to  $u_4$  then  $\{u_5, u_3, v_2, v_3, z\}$  is a TRDS of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq 6 < \frac{4.12}{7} = \frac{4n}{7}$  which is a contradiction. If  $\mathcal{U} = C_3$  then  $n(G) = 10$ . The set  $\{u_1, v_1, v_2, v_3, z\}$  is a TRDS of  $G$  and so  $\gamma_{\text{tr}}(G) \leq 5 < \frac{4.10}{7} = \frac{4n}{7}$ , which is a contradiction.

Case 1.2.2.  $N(v_2) \subseteq \{v_1, v_3\} \cup V(\mathcal{U})$ .

Let  $G'_1 = \langle V(P) \cup V(\mathcal{U}) \rangle$  and  $G'_2 = G - G'_1$ . Note that  $G'_1$  is a necklace. Since  $G$  is connected we have that  $G'_2$  has a vertex that is adjacent to either  $v_3$  or  $v_4$ . If a vertex  $z \in V(G'_2)$  is adjacent to  $v_3$  then since  $\langle \{z, v_2, v_4, v_3\} \rangle$  is not a claw we have that  $v_4$  is adjacent to  $z$ . This fact implies that  $v_4$  has a neighbor in  $V(G'_2)$  and since  $\deg(v_4) = 2$ , we have that  $G'_2$  is connected. Since  $\delta(G_2) \geq 2$  we have that  $\delta(G'_2) \geq 2$ . If  $G'_2 \notin \mathcal{K}$  then, by [Observation 6](#), we are done.

If  $G'_2 \in \mathcal{K}$  then since  $\deg(v_4) = 2$ , we have, by [Observation 8](#), that  $G'_2 \in \{C_3, \mathcal{B}\}$ . Clearly,  $G_2$  consists out of two components which are both in the set  $\{C_3, \mathcal{B}\}$ . By [Observation 9](#) we get a contradiction.

Case 2. The vertex  $v_1$  is not adjacent to a vertex in  $V(\mathcal{U})$ .

By symmetry, we have that  $v_4$  is not adjacent to any vertex in  $V(\mathcal{U})$ . Since  $G$  is connected, we have, without loss of generality, that  $v_2$  is adjacent to some vertex in  $V(\mathcal{U})$ . By [Observation 8](#),  $v_2$  is adjacent to say  $u_1$ . Define  $G'_1 = \langle \{v_1, v_2, v_3, v_4\} \cup V(\mathcal{U}) \rangle$  and  $G'_2 = G - G'_1$ . Note that if  $V(G'_2) \neq \emptyset$  then since  $\delta(G_2) \geq 2$  we have that  $\delta(G'_2) \geq 2$ .

If  $G'_2$  has a component  $\mathcal{U}' \in \mathcal{K}$  then  $\mathcal{U}'$  is also a component of  $G_2$ , and, by Case 1, every vertex of  $\mathcal{U}'$  is adjacent to neither  $v_1$  nor  $v_4$ . Since  $G$  is connected, we must have that some  $z$  in  $V(\mathcal{U}')$  is adjacent to either  $v_2$  or  $v_3$ . If  $z$  is adjacent to  $v_2$  then  $\langle \{z, v_1, u_1, v_2\} \rangle$  is a claw, a contradiction. Hence,  $z$  is adjacent to  $v_3$  and no vertex of  $\mathcal{U}'$  is adjacent to  $v_2$ . If  $v_1$  is adjacent to  $v_3$  then  $\langle \{z, v_4, v_1, v_3\} \rangle$  is a claw, a contradiction. Hence,  $v_1$  is not adjacent to  $v_3$ . The fact that  $\langle \{u_1, v_3, v_1, v_2\} \rangle$  is not a claw implies that  $v_3$  is adjacent to  $u_1$  and so  $\langle \{u_1, v_4, z, v_3\} \rangle$  is a claw, a contradiction.

We may assume that either  $V(G'_2) = \emptyset$ , or  $V(G'_2) \neq \emptyset$  with  $G'_2$  not having any component in  $\mathcal{K}$ . In both cases  $G'_2$  has a TRDS  $S$  of cardinality at most  $\frac{4n(G'_2)}{7}$ .

Case 2.1.  $v_1$  is adjacent to  $v_3$ .

Clearly,  $G'_1$  is a necklace with initial vertex  $v_4$  and end vertex  $v_2$ . By [Observation 6](#) we get a contradiction.

Case 2.2.  $v_1$  and  $v_3$  are not adjacent.

Since  $\{u_1, v_3, v_1, v_2\}$  is not a claw we have that  $u_1$  and  $v_3$  are adjacent. By symmetry,  $v_4$  is not adjacent to  $v_2$ . By Claim 1 we have that  $\mathcal{U} \in \{C_3, \mathcal{B}\}$ . Suppose that  $\mathcal{U} = \mathcal{B}$  ( $\mathcal{U} = C_3$  respectively). If  $v_1$  or  $v_4$ , say  $v_1$ , is adjacent to a vertex in  $S$  then  $S \cup \{v_3, v_4, u_2, u_3\}$  ( $S \cup \{v_3, v_4, u_1\}$  respectively) is a TRDS of  $G$ . If  $v_1$  is adjacent to  $v_4$  or  $v_1$  and  $v_4$  both have neighbors in  $V(G'_2) - S$ , then  $S \cup \{v_2, v_3, u_1, u_2, u_3\}$  ( $S \cup \{v_2, v_3, u_1\}$  respectively) is a TRDS of  $G$ . Hence, if  $\mathcal{U} = \mathcal{B}$  then  $\gamma_{\text{tr}}(G) \leq 5 + |S| < \frac{4n}{7} - \frac{36}{7} + \frac{35}{7} < \frac{4n}{7}$  and if  $\mathcal{U} = C_3$  then  $\gamma_{\text{tr}}(G) \leq 3 + |S| < \frac{4n}{7} - \frac{28}{7} + \frac{21}{7} < \frac{4n}{7}$ . In both cases, we get a contradiction.  $\square$

**Proof of Lemma 14.** Let  $P : v_1, v_2, v_3, v_4$  be a path of which the vertices  $v_1$  and  $v_4$  have degree at least two and  $\deg(v_3) = 2$  and define  $G_1 = \langle \{v_1, v_2, v_3, v_4\} \rangle$  and  $G_2 = G - G_1$ . If  $V(G_2) = \emptyset$  then  $\gamma_{\text{tr}}(G) \leq \frac{4n}{7}$ , a contradiction. We may assume, to the contrary, that  $\delta(G_2) \geq 2$ . If  $G_2$  has no component in  $\mathcal{K}$  then, by [Observation 7](#), we have that  $G$  has a TRDS of cardinality at most  $\frac{4n}{7}$ , a contradiction. Hence,  $G_2$  has a component  $\mathcal{U}$  in  $\mathcal{K}$ .

**Claim 1.** Any component in  $G_2$  has a vertex that is adjacent to either  $v_1$  or  $v_4$ .

**Proof.** Let  $D$  be a component of  $G_2$ . Since  $G$  is connected there is some vertex  $v \in V(D)$  adjacent to either  $v_1, v_4$  or  $v_2$ . If  $v$  is adjacent to say  $v_2$  then since  $\{v, v_3, v_1, v_2\}$  is not a claw we must have that  $v$  is adjacent to  $v_1$  and so we are done.  $\square$

Case 1.  $V(G_2) - V(\mathcal{U}) = \emptyset$ .

In this case  $G_2 = \mathcal{U}$ . By Claim 1 and [Observation 8](#), the graph  $G$  is a necklace with attachment that has length three. This contradicts [Observation 6](#).

Case 2.  $V(G_2) - V(\mathcal{U}) \neq \emptyset$ .

Case 2.1.  $v_4$  is adjacent to say  $u_1$ .

If  $v_4$  is adjacent to a vertex  $v \in V(G_2) - V(\mathcal{U})$ , then  $\{v, u_1, v_3, v_4\}$  is a claw which is a contradiction. Hence,  $N(v_4) \subset V(\mathcal{U}) \cup V(P)$ . Note that if  $\mathcal{U} \notin \{C_3, \mathcal{B}\}$  then  $v_4$  is adjacent to say  $u_2$ . Clearly, the path  $P$  together with  $\mathcal{U}$  forms a necklace and so we define  $G'_1 = \langle V(P) \cup V(\mathcal{U}) \rangle$  and  $G'_2 = G - G'_1$ . If  $V(G'_2) = \emptyset$  then we obtain a contradiction by [Observation 6](#). Now clearly  $\delta(G'_2) \geq 2$  and if  $G'_2$  has no components in  $\mathcal{K}$  then we get, by [Observation 6](#), a contradiction. Hence,  $G'_2$  has a component  $\mathcal{U}'$  in  $\mathcal{K}$ . Since  $N(v_4) \subset V(\mathcal{U}) \cup V(P)$ , we have, by Claim 1, that  $v_1$  must be adjacent to a vertex of  $\mathcal{U}'$ .

Suppose that either  $V(G'_2) - V(\mathcal{U}') \neq \emptyset$  or  $v_1$  is adjacent to two vertices of  $\mathcal{U}'$ . The path  $P - v_1$  together with  $\mathcal{U}$  forms a necklace and so we define  $G''_1 = \langle V(P) \cup V(\mathcal{U}) - \{v_1\} \rangle$  and  $G''_2 = G - G''_1$ . If  $v_1$  is adjacent to two vertices of  $\mathcal{U}'$  then obviously  $\deg(v_1) \geq 3$ . If  $V(G'_2) - V(\mathcal{U}') \neq \emptyset$  then since  $N(v_4) \subset V(\mathcal{U}) \cup V(P)$ , we have, by Claim 1, that there is a vertex  $v \in V(G'_2) - V(\mathcal{U}')$  adjacent to  $v_1$ . Hence,  $\deg(v_1) \geq 3$ . It follows that  $\delta(G''_2) \geq 2$ . Since  $N(v_4) \subset V(\mathcal{U}) \cup V(P)$  we have, by Claim 1, that  $G''_2$  is connected and  $G''_2 \notin \mathcal{K}$ . By [Observation 6](#) we are done.

Hence,  $V(G'_2) - V(\mathcal{U}') = \emptyset$  and  $v_1$  is adjacent to exactly one vertex of  $\mathcal{U}'$ . The first part of [Observation 8](#) must imply that  $\mathcal{U}' \in \{C_3, \mathcal{B}\}$ . The path  $P - v_4$  together with  $\mathcal{U}'$  forms a necklace and so we define  $G''_1 = \langle V(P) \cup V(\mathcal{U}') - \{v_4\} \rangle$  and  $G''_2 = G - G''_1$ . If  $v_4$  is adjacent to two vertices of  $\mathcal{U}$  then  $\delta(G''_2) \geq 2$ . Furthermore,  $G''_2 \notin \mathcal{K}$ . By [Observation 6](#) we get a contradiction. Therefore,  $v_4$  is adjacent to exactly one vertex of  $\mathcal{U}$ . The first part of [Observation 8](#) must imply that  $\mathcal{U} \in \{C_3, \mathcal{B}\}$ .

It follows that  $V(G_2) = V(\mathcal{U}') \cup V(\mathcal{U})$  and that  $v_1$  ( $v_4$  respectively) is adjacent to exactly one vertex of  $\mathcal{U}'$  ( $\mathcal{U}$  respectively). By [Observation 9](#) we get a contradiction.

Case 2.2.  $v_1$  is adjacent to  $u_1$ .

Suppose that  $\mathcal{U}'$  is a component of  $G_2$ , with  $\mathcal{U}' \in \mathcal{K}$ . If  $\mathcal{U}' \neq \mathcal{U}$  and  $v_4$  is adjacent to a vertex of  $\mathcal{U}'$ , then we may re-label  $\mathcal{U}'$  as  $\mathcal{U}$  and so by the previous case we are done. It follows that if  $G_2$  has a component in  $\mathcal{K}$ , then no vertex of this component can be adjacent to  $v_4$ . Note that the path  $P$  together with  $\mathcal{U}$  is a necklace. Let  $G'_1 = \langle V(P) \cup V(\mathcal{U}) \rangle$  and  $G'_2 = G - G'_1$ . If  $V(G'_2) = \emptyset$  then we obtain a contradiction by [Observation 6](#). Since  $\delta(G_2) \geq 2$ , we may assume that  $\delta(G'_2) \geq 2$ . If  $G'_2$  has no components in  $\mathcal{K}$ , then, by [Observation 6](#), we get a contradiction. Hence,  $G'_2$  has a component  $\mathcal{U}'$  in  $\mathcal{K}$ .

Note that  $\mathcal{U}'$  is also a component of  $G_2$ . By Claim 1 and the fact that  $v_4$  is not adjacent to any vertex of  $\mathcal{U}'$ , we have that  $v_1$  is adjacent to a vertex  $v$  of  $\mathcal{U}'$ . In addition,  $N(v_4) \subseteq V(G) - \{v_4\} - V(\mathcal{U}) - V(\mathcal{U}')$ . Since  $\{v, u_1, v_2, v_1\}$  is not a claw we have that  $v_2$  is adjacent to  $v$  or  $v_2$  is adjacent to  $u_1$ . We also claim that  $v_1$  is not adjacent to a vertex  $w \in V(G_2) - V(\mathcal{U}) - V(\mathcal{U}')$ . Suppose, to the contrary, that  $v_1$  is adjacent to  $w$ . Then  $\{w, v, u_1, v_1\}$  induces a claw which is a contradiction. Also, if  $v_4$  is adjacent to  $v_1$  then  $\{u_1, v, v_4, v_1\}$  induces a claw. Hence,  $N(v_1) \subseteq \{v_2\} \cup V(\mathcal{U}) \cup V(\mathcal{U}')$  and  $N(v_4) \subseteq \{v_3, v_2\} \cup V(G_2) - V(\mathcal{U}) - V(\mathcal{U}')$ .

If  $v_2$  is adjacent to  $v$ , then let  $G''_1 = \langle \{v_1\} \cup V(\mathcal{U}) \rangle$ . If  $v_2$  is adjacent to  $u_1$ , then re-label  $\mathcal{U}'$  as  $\mathcal{U}$  and  $\mathcal{U}$  as  $\mathcal{U}'$  and set  $G''_1 = \langle \{v_1\} \cup V(\mathcal{U}) \rangle$ . Define  $G''_2 = G - G''_1$ . Note that  $\delta(G''_2) \geq 2$ . The fact that  $N(v_1) \subseteq \{v_2\} \cup V(\mathcal{U}) \cup V(\mathcal{U}')$  implies that  $G''_2$  is connected. Clearly,  $G''_2 \notin \mathcal{K}$ . If  $\mathcal{U} = C_3$  ( $\mathcal{U} = \mathcal{B}$  respectively) then by [Observation 7](#) (second part of [Observation 10](#) respectively) we are done. Hence,  $\mathcal{U} \notin \{C_3, \mathcal{B}\}$ . By the first part of [Observation 10](#) we get a contradiction.  $\square$



**Proof of Lemma 15.** Suppose, to the contrary, that  $G$  has a 2-path  $P : x, v_1, v_2, y$ . Let  $G_1 = \langle V(P) \rangle$  and  $G_2 = G - G_1$ . Since  $x$  and  $y$  both have degree greater than two and  $\deg(v_2) = 2$ , we may apply Lemma 14 to deduce that  $\delta(G_2) \leq 1$ . Let  $v$  be a vertex of  $G_2$ , with  $\deg_{G_2}(v) \leq 1$ .

Case 1.  $\deg_{G_2}(v) = 0$ .

Obviously,  $N(v) = \{x, y\}$ . If  $x$  is adjacent to a vertex  $w$  in  $V(G) - V(P) - \{v\}$ , then  $\langle \{v, w, v_1, x\} \rangle$  is a claw which is a contradiction. Hence,  $N(x) = \{v_1, v, y\}$ . We can show, in similar fashion, that  $N(y) = \{v_2, v, x\}$ . Hence,  $n(G) = 5$  and  $\{x, v_1\}$  is a TRDS of  $G$  and so  $\gamma_{tr}(G) \leq 2 \leq \frac{4.5}{7} = \frac{4n}{7}$  which is a contradiction.

Case 2.  $\deg_{G_2}(v) = 1$ .

Case 2.1. The vertex  $v$  is adjacent to both  $x$  and  $y$ .

Since  $\deg_{G_2}(v) = 1$  we must have that  $\deg(v) = 3$ .

Case 2.1.1.  $x$  is adjacent to a vertex  $w \in V(G) - V(P) - \{v\}$ .

Since  $\langle N[x] - \{v_1\} \rangle$  is complete, we must have that  $v$  is adjacent to  $w$  and so  $N(v) = \{x, y, w\}$ . Furthermore,  $\{v, w, v_1\} \subseteq N(x) \subseteq \{v, w, v_1, y\}$ . Since  $\langle N[y] - \{v_2\} \rangle$  is complete and  $\deg(v) = 3$ , we can conclude that  $N(y) \cap (V(G) - V(P) - \{v, w\}) = \emptyset$ . Hence,  $\{v, v_2\} \subset N(y) \subseteq \{v, v_2, w, x\}$ . We may form  $G'$  by deleting from  $G$  the edge  $yw$  (if it exists) and  $xy$  (if it exists). Note that  $\delta(G') \geq 2$  and  $G' \notin \mathcal{K}$ . If  $G'$  is a claw-free graph then  $G'$  has smaller size than  $G$  and so it follows that  $G'$  has a TRDS  $S'$  of cardinality at most  $\frac{4n(G')}{7}$ . Clearly,  $S'$  is a TRDS of  $G$  and so  $\gamma_{tr}(G) \leq |S'| \leq \frac{4n}{7}$  which is a contradiction. Hence,  $G'$  has a claw  $\langle \{w_4, w_3, w_2, w_1\} \rangle$ . Note that, without loss of generality,  $w_2 = y$  and  $w_3 \in \{x, w\}$ .

If  $w_3 = x$  then the fact that  $\{v, v_2\} \subset N(y) \subseteq \{v, v_2, w, x\}$  implies that  $w_1 = v$ . But since  $N(v) = \{x, y, w\}$ , we have that  $w_4 = w$ . Hence,  $w_4$  is adjacent to  $w_3$  which is a contradiction. Hence,  $w_3 = w$ . It follows once more that  $w_1 = v$  and so  $w_4 = x$ . But then  $w_4$  and  $w_3$  are adjacent which is a contradiction.

Case 2.1.2.  $N(x) = \{v_1, y, v\}$  and  $N(y) = \{v_2, x, v\}$ .

Let  $w_1 \in N(v) - \{x, y\}$ . Suppose first that  $\deg(w_1) \geq 3$ . Let  $G'_1 = \langle V(P) \cup \{v\} \rangle$  and we consider the connected claw-free graph  $G'_2 = G - G'_1$ . Now  $G'_2$  has minimum degree at least two. If  $G'_2 \notin \mathcal{K}$  then  $G'_2$  has a TRDS  $S_2$  with cardinality at most  $\frac{4(n-5)}{7}$ . The set  $\{x, v_1\} \cup S_2$  is a TRDS of  $G$  and so  $\gamma_{tr}(G) \leq |S_2| + 2 \leq \frac{4n}{7} - \frac{20}{7} + \frac{14}{7} \leq \frac{4n}{7}$ , a contradiction. Hence,  $G'_2 \in \mathcal{K}$ . By the first part of Observation 8, and the fact that  $\deg(v) = 3$ , we can conclude that  $G'_2 \in \{C_3, \mathcal{B}\}$ . By Observation 8 we may also assume that  $v$  is adjacent to say  $u_1$ .

If  $G'_2 = C_3$  then  $n(G) = 8$  and  $\{v, u_1, x, v_1\}$  is a TRDS of  $G$  and so  $\gamma_{tr}(G) \leq 4 < \frac{4.8}{7} = \frac{4n}{7}$ , which is a contradiction.

If  $G'_2 = \mathcal{B}$  then we form  $G' = G - u_1u_3$ . The graph  $G'$  is connected and claw-free. By Observation 11, we are done.

Hence,  $\deg(w_1) = 2$ . Let  $w_2 \in N(w_1) - \{v\}$ . We have that there exists a 2-path  $P' : v, w_1, w_2, \dots, w_j, w_{j+1}$ , where  $w_i \in V(G) - V(P) - \{v\}$  for  $i = 1, 2, \dots, j + 1$ . By Lemma 12 we have that  $j = 1$  or  $j = 2$ .

Case 2.1.2.1.  $j = 2$ .

We let  $G''_1 = \langle V(P') \cup V(P) - \{w_3\} \rangle$  and  $G''_2 = G - G''_1$ . If  $G''_2 \notin \mathcal{K}$  then  $G''_2$  has a TRDS  $S_2$  with cardinality at most  $\frac{4(n-7)}{7}$ . If  $w_3 \in S_2$  then  $\{v, x, v_1\} \cup S_2$  is a TRDS of  $G$ . If  $w_3 \notin S_2$  then  $\{v, w_1, x, v_1\} \cup S_2$  is a TRDS of  $G$ . Hence,  $\gamma_{tr}(G) \leq |S_2| + 4 \leq \frac{4n}{7} - \frac{28}{7} + \frac{28}{7} \leq \frac{4n}{7}$  which is a contradiction. We may therefore conclude that  $G''_2 \in \mathcal{K}$ .

By the first part of Observation 8, and the fact that  $\deg(w_2) = 2$ , we can deduce that  $G''_2 \in \{C_3, \mathcal{B}\}$ . By Observation 8 we may also assume that say  $u_1 = w_3$ .

If  $G''_2 = C_3$  then  $n(G) = 10$  and  $\{u_1, w_2, x, v_1\}$  is a TRDS of  $G$  and so  $\gamma_{tr}(G) \leq 4 \leq \frac{4.10}{7} = \frac{4n}{7}$ , a contradiction.

If  $G''_2 = \mathcal{B}$  then we form  $G' = G - u_1u_3$ . The graph  $G'$  is connected and claw-free. By Observation 11, we obtain a contradiction.

Case 2.1.2.2.  $j = 1$ .

We let  $G''_1 = \langle V(P) \cup \{v, w_1\} \rangle$  and  $G''_2 = G - G''_1$ . Note that  $G''_2$  is claw-free, connected and has minimum degree at least two. If  $G''_2 \notin \mathcal{K}$  then  $G''_2$  has a TRDS  $S_2$  with cardinality at most  $\frac{4(n-6)}{7}$ . If  $w_2 \notin S_2$  then  $\{v, x, v_1\} \cup S_2$  is a TRDS of  $G$ . If  $w_2 \in S_2$  then  $\{x, v_1\} \cup S_2$  is a TRDS of  $G$ . Hence,  $\gamma_{tr}(G) \leq |S_2| + 3 \leq \frac{4n}{7} - \frac{24}{7} + \frac{21}{7} < \frac{4n}{7}$  which is a contradiction. Hence,  $G''_2 \in \mathcal{K}$ .

By the first part of Observation 8, and the fact that  $\deg(w_1) = 2$ , we can deduce that  $G''_2 \in \{C_3, \mathcal{B}\}$ . By Observation 8 we may also assume that say  $u_1 = w_2$ .

If  $G''_2 = C_3$  then  $n(G) = 9$  and  $\{u_1, w_1, x, v_1\}$  is a TRDS of  $G$  and so  $\gamma_{tr}(G) \leq 4 < \frac{4.9}{7} = \frac{4n}{7}$ .

If  $G''_2 = \mathcal{B}$  then we form  $G' = G - u_1u_3$ . The graph  $G'$  is connected and claw-free. By Observation 11, we obtain a contradiction.

Case 2.2. The vertex  $v$  is adjacent to only  $x$ .

Clearly,  $\deg(v) = 2$ . If  $x$  is adjacent to  $y$  then, since  $\langle N(x) - \{v_1\} \rangle$  is complete, we must have that  $v$  is adjacent to  $y$ , which is a contradiction. Hence,  $x$  and  $y$  are not adjacent. It follows, since  $\deg(x) \geq 3$ , that there is a vertex  $w$  in  $N(x) \cap (V(G) - V(P) - \{v\})$ . The completeness of  $\langle N(x) - \{v_1\} \rangle$  also implies that  $N(x) = \{v_1, v, w\}$ . If  $\deg(w) \geq 3$  then the graph  $G' = G - xw$  is claw-free and connected. By Observation 11, we get a contradiction. So  $\deg(w) = 2$ .

If  $y$  has a degree two neighbor  $v' \in V(G) - V(P) - \{v, w\}$  then, by the same argument, there is a vertex  $w' \in V(G) - V(P) - \{v, w, v'\}$ , such that  $N(y) = \{v_2, v', w'\}$  and  $\deg(w') = 2$ . Hence,  $n(G) = 8$ . The set  $\{x, v_1, v_2, y\}$  is a TRDS of  $G$  and so  $\gamma_{tr}(G) \leq 4 < \frac{4.8}{7} = \frac{4n}{7}$ , a contradiction.

We may assume that the only degree two vertex adjacent to  $y$  is  $v_2$ . Note that  $G'_1 = \langle V(P) \cup N(x) \rangle$  induces a necklace with attachment  $P - x$ . Let  $G'_2 = G - G'_1$  and note that since  $G$  is claw-free we have that  $G'_2$  is connected. Also,  $G'_2$  has minimum degree at least two. If  $G'_2 \notin \mathcal{K}$  then, by [Observation 6](#), we have a contradiction. Hence,  $G'_2 \in \mathcal{K}$ .

Suppose that  $G'_2 \in \mathcal{K} - \{\mathcal{B}\}$ . We may assume that  $y$  is adjacent to say  $u_1$ . If  $i \geq 5$  then we have, since  $N(y) - \{v_2\}$  induces a clique, that  $\deg(y) = 3$ . Furthermore, by the first part of [Observation 8](#),  $y$  is adjacent to say  $u_1$  and  $u_2$ . Also,  $u_2, \dots, u_i, u_1$  is a 2-path of length greater than or equal to four. This will contradict [Lemma 12](#). Hence,  $i = 3$  and so  $n(G) = 9$ . The set  $\{x, v_1, v_2, y, u_1\}$  is a TRDS of  $G$  and so  $\gamma_{\text{tr}}(G) \leq 5 < \frac{4.9}{7} = \frac{4n}{7}$ .

If  $G'_2 = \mathcal{B}$  then  $n(G) = 11$ . By the second part of [Observation 8](#),  $y$  is adjacent to  $u_1$ . The set  $\{u_1, u_2, u_3, x, v_1\}$  is a TRDS of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq 5 < \frac{4.11}{7} = \frac{4n}{7}$ .  $\square$

**Proof of Lemma 16.** If  $G$  has a 2-path of length greater than or equal to two, then, by [Lemmas 12](#) and [15](#), we are done. Hence, we may assume that  $G$  has no 2-path. Suppose, to the contrary, that  $G$  has a 2-cycle  $P : x, v_1, v_2, \dots, v_{j+1} = x$ . Let  $w \in N(x) - \{v_1, v_j\}$ .

*Claim 1.* There are no 2-cycles on more than three vertices.

**Proof.** For the 2-cycle  $P : x, v_1, v_2, \dots, v_{j+1} = x$ , suppose, to the contrary, that  $j \geq 3$ . Then  $\langle \{w, v_1, v_j, x\} \rangle$  is a claw, a contradiction. Hence,  $j = 2$ .  $\square$

*Claim 2.* No vertex in  $N(x) - \{v_1, v_2\}$  has degree two.

**Proof.** Suppose, to the contrary, that a vertex  $v \in N(x) - \{v_1, v_2\}$  has degree two. Let  $N(v) = \{x, v'\}$ . By Claim 1 and the fact that  $G$  has no 2-path,  $\deg(v') = 2$  and  $N(v') = \{x, v\}$ . Since  $G$  is claw-free we have that  $\deg(x) = 4$  and so  $G = \mathcal{B}$ , a contradiction.  $\square$

*Case 1.*  $\deg(x) = 3$ .

By Claim 2,  $\deg(w) \geq 3$ . Let  $w', w'' \in N(w) - \{x\}$ . Since  $G$  is claw-free,  $w'$  is adjacent to  $w''$ . Suppose first that  $\deg(w') = 2$ . If  $\deg(w'') \geq 3$  then  $ww'w''$  is a 2-path, a contradiction. Hence,  $\deg(w'') = 2$ . It follows that  $n(G) = 6$  and the set  $\{x, w\}$  is a TRDS of  $G$  and so  $\gamma_{\text{tr}}(G) \leq 2 < \frac{4.6}{7} = \frac{4n}{7}$ . Hence,  $\deg(w') \geq 3$  and by symmetry we have that  $\deg(w'') \geq 3$ . We now define  $G_1 = \langle \{x, v_1, v_2, w\} \rangle$  and  $G_2 = G - G_1$ . Clearly,  $\delta(G_2) \geq 2$  and  $G_2$  is claw-free and connected. If  $G_2 \notin \mathcal{K}$  then, by [Observation 7](#), we get a contradiction. Hence,  $G_2 \in \mathcal{K}$ .

If  $G_2 \in \mathcal{K} - \{C_3, \mathcal{B}\}$  then  $G_2$  has a 2-path of length greater than three which contradicts [Lemma 12](#).

If  $G_2 = C_3$  then  $n(G) = 7$  and  $\{x, w, w'\}$  is a TRDS of  $G$  and so  $\gamma_{\text{tr}}(G) \leq 3 < \frac{4.7}{7} = \frac{4n}{7}$ , which is a contradiction.

If  $G_2 = \mathcal{B}$  then  $n(G) = 9$ . Clearly,  $G_2$  has a TRDS  $S$  of cardinality at most three. The set  $S \cup \{x, w\}$  is a TRDS of  $G$  and so  $\gamma_{\text{tr}}(G) \leq 5 < \frac{4.9}{7} = \frac{4n}{7}$ , which is a contradiction.

*Case 2.*  $\deg(x) \geq 4$ .

*Case 2.1.* The vertex  $w$  is adjacent to a degree two vertex  $v \in V(G) - N[x]$ .

Let  $v' \in N(v) - \{w\}$ . If  $\deg(v') \geq 3$  then  $G$  has a 2-path of length two, which is a contradiction. Hence,  $v'$  has degree two. To avoid a 2-path of length greater than or equal to two, or a 2-cycle on more than three vertices, we must have that  $N(v') = \{v, w\}$ . If  $w$  is adjacent to a vertex  $w' \in V(G) - N[x] - \{v', v\}$ , then  $\langle \{w', v, x, w\} \rangle$  will induce a claw which is a contradiction. Hence,  $N[w] = \{v, v'\} \cup (N[x] - \{v_1, v_2\})$  and so  $N[w] - \{v, v'\} = N[x] - \{v_1, v_2\}$ . Let  $G_1 = \langle \{x, v_1, v_2, w, v, v'\} \rangle$  and  $G_2 = G - G_1$ . Since  $\langle N(x) - \{v_1, v_2\} \rangle$  is complete, we must have that  $G_2$  is connected.

*Case 2.1.1.*  $\delta(G_2) \leq 1$ .

Let  $w' \in N(x) - \{v_1, v_2, w\}$ . We may assume, without loss of generality, that  $\deg_{G_2}(w') \leq 1$ . Since  $\langle N(x) - \{v_1, v_2\} \rangle$  is complete, we have that  $N[x] - \{v_1, v_2, w'\} \subseteq N(w')$ . In addition, by Claim 2, we have that  $\deg_{G_2}(w') = 1$  and so  $\deg(w') = 3$ .

*Case 2.1.1.1.*  $w'$  has a neighbor  $z \in V(G) - N[x] - \{v, v'\}$ .

Note that, since  $\langle N(x) - \{v_1, v_2\} \rangle$  is complete and  $N[w] - \{v, v'\} = N[x] - \{v_1, v_2\}$ , we have that  $N(w') = \{w, x, z\}$ ,  $N(x) = \{v_1, v_2, w, w'\}$  and  $N(w) = \{v, v', w', x\}$ . We let  $G'_1 = \langle N[x] \cup \{v, v'\} \rangle$  and  $G'_2 = G - G'_1$ . Clearly,  $G'_2$  is connected.

*Case 2.1.1.1.1.*  $\delta(G'_2) \geq 2$ .

If  $G'_2 \notin \mathcal{K}$ , then note that  $G'_2$  has smaller size than  $G$  and so  $G'_2$  has a TRDS  $S_2$  of cardinality at most  $\frac{4(n-7)}{7}$ . The set  $S_2 \cup \{x, w, w'\}$  is a TRDS of  $G$  and so  $\gamma_{\text{tr}}(G) \leq |S_2| + 3 \leq \frac{4n}{7} - \frac{28}{7} + \frac{21}{7} < \frac{4n}{7}$ , which is a contradiction. Hence,  $G'_2 \in \mathcal{K}$ .

If  $G'_2 \in \mathcal{K}$  then, without loss of generality,  $z = u_1$ . If  $G'_2 \in \mathcal{K} - \{C_3, \mathcal{B}\}$  then, since  $N(w') = \{w, x, z\}$ , the first part of [Observation 8](#) produces a contradiction.

If  $G'_2 = C_3$  then  $n(G) = 10$ . The set  $\{x, w, w', u_1\}$  is a TRDS of  $G$  and so  $\gamma_{\text{tr}}(G) \leq 4 < \frac{4.10}{7} = \frac{4n}{7}$ , which is a contradiction.

If  $G'_2 = \mathcal{B}$  then  $n(G) = 12$ . The set  $\{x, w, w', u_1, u_2, u_3\}$  is a TRDS of  $G$ . Hence  $\gamma_{\text{tr}}(G) \leq 6 < \frac{4.12}{7} = \frac{4n}{7}$ , which is a contradiction.

*Case 2.1.1.1.2.*  $\delta(G'_2) \leq 1$ .

In this case we may assume that the vertex  $z$  has degree one in  $G'_2$ . Since  $\deg(w') = 3$ , we have that  $G$  has a 2-path of length at least two, a contradiction.

Case 2.1.1.2.  $w'$  has no neighbor in  $V(G) - N[x] - \{v, v'\}$ .

Since  $\langle N(x) - \{v_1, v_2\} \rangle$  is a clique and  $\deg(w') = 3$ , we have that  $\deg(x) = 5$ . Let  $w'' \in N(x) - \{v_1, v_2, w, w'\}$ . If  $\deg(w'') = 3$  then  $N(w'') = \{x, w, w'\}$  and  $n(G) = 8$ . The set  $\{v_1, v_2, x, w\}$  is a *TRDS* of  $G$  and so  $\gamma_{\text{tr}}(G) \leq 4 < \frac{4.8}{7} = \frac{4n}{7}$ , which is a contradiction. Therefore,  $\deg(w'') \geq 4$  and so  $w''$  is adjacent to at least one vertex in  $V(G) - N[x] - \{v, v'\}$ . Note that the path  $P' : w', x, v_1, v_2$  is such that  $w'$  and  $v_2$  both have degree at least two and  $\deg(v_1) = 2$ . We form  $G'_1 = \langle \{x, w', v_1, v_2\} \rangle$  and  $G'_2 = G - G'_1$ . Note that  $\delta(G'_2) \geq 2$  and so we obtain a contradiction by Lemma 14.

Case 2.1.2.  $\delta(G_2) \geq 2$ .

If  $G_2 \notin \mathcal{K}$  then it follows that  $G_2$  has a *TRDS*  $S_2$  of cardinality at most  $\frac{4(n-6)}{7}$ . The set  $S_2 \cup \{x, w\}$  is a *TRDS* of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S_2| + 2 \leq \frac{4n}{7} - \frac{24}{7} + \frac{14}{7} < \frac{4n}{7}$ . Thus,  $G_2 \in \mathcal{K}$ .

Suppose that  $G_2 \in \mathcal{K} - \{C_3, \mathcal{B}\}$ . By the first part of Observation 8, and the fact that  $N[w] - \{v, v'\} = N[x] - \{v_1, v_2\}$ , we have that  $w$  and  $x$  are both adjacent to say  $u_1$  and  $u_2$ . Also note that if  $u_3$  has degree two, then  $G$  will have a 2-path of length at least two, a contradiction. Hence, both  $x$  and  $w$  are adjacent to  $u_3$ . Since  $\langle N(x) - \{v_1, v_2\} \rangle$  is a clique, we have that  $u_1$  and  $u_3$  are adjacent. This is a contradiction.

If  $G_2 = C_3$  then  $w$  (and  $x$ ) are both adjacent to say  $u_1$ . The set  $\{x, w, u_1\}$  is a *TRDS* of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq 3 < \frac{4.9}{7} = \frac{4n}{7}$  which is a contradiction.

If  $G_2 = \mathcal{B}$  then  $n(G) = 11$  and the set  $\{x, w, u_1, u_2, u_3\}$  is a *TRDS* of  $G$ . Hence  $\gamma_{\text{tr}}(G) \leq 5 < \frac{4.11}{7} = \frac{4n}{7}$ , which is a contradiction.

Case 2.2. No vertex in  $N(x) - \{v_1, v_2\}$  has a degree two neighbor in  $V(G) - N[x]$ .

Let  $G_1 = \langle V(P) \cup \{w\} \rangle$  and  $G_2 = G - G_1$ . Note that the path  $P' : w, x, v_1, v_2$  is such that  $w$  and  $v_2$  both have degree at least two and  $\deg(v_1) = 2$ . By Lemma 14 it follows that  $\delta(G_2) \leq 1$ . Since no vertex in  $N(x) - \{v_1, v_2\}$  has a degree two neighbor in  $V(G) - N[x]$ , we have that any vertex of degree at most one in  $G_2$  must be adjacent to  $x$ . Furthermore, Claim 2 implies that  $\delta(G_2) = 1$ . Let  $w'$  be a degree one vertex of  $G_2$ . Note that  $\deg(w') = 3$ .

Case 2.2.1.  $N(x) - \{w, w', v_1, v_2\} \neq \emptyset$ .

Let  $w'' \in N(x) - \{v_1, v_2, w, w'\}$ . Hence,  $N(w') = \{w, x, w''\}$  and  $\deg(x) = 5$ . Furthermore,  $N(x) = \{v_1, v_2, w', w, w''\}$ .

Case 2.2.1.1.  $\deg(w) = 3$ .

Clearly,  $N(w) = \{w', x, w''\}$ . If  $\deg(w'') = 3$  then  $n(G) = 6$  and the set  $\{x, v_1, v_2\}$  is a *TRDS* of  $G$ . Therefore,  $\gamma_{\text{tr}}(G) \leq 3 < \frac{4.6}{7} = \frac{4n}{7}$  which is a contradiction. Hence,  $\deg(w'') \geq 4$ . We form the graph  $G'$  by removing, from  $G$ , the edges  $xw''$  and  $ww''$ . The graph  $G'$  is claw-free, connected and has minimum degree at least two. Also,  $G' \notin \mathcal{K}$  and so  $G'$  has a *TRDS*  $S$  of cardinality at most  $\frac{4n}{7}$ . Since  $S$  is also a *TRDS* of  $G$  we have that  $\gamma_{\text{tr}}(G) \leq |S| \leq \frac{4n}{7}$ , which is a contradiction.

Case 2.2.1.2.  $\deg(w) \geq 4$ .

By symmetry,  $\deg(w'') \geq 4$ . We re-label  $w'$  as  $w$  and so  $\delta(G_2) \geq 2$ , which is impossible by Lemma 14.

Case 2.2.2.  $N(x) - \{w, w', v_1, v_2\} = \emptyset$ .

Hence,  $\deg(x) = 4$ . If  $\deg(w) \geq 4$  then we may re-label  $w'$  as  $w$  and so  $\delta(G_2) \geq 2$ , which is impossible by Lemma 14. Hence,  $\deg(w) = 3$ . We let  $G'_1 = \langle N[x] \rangle$  and  $G'_2 = G - G'_1$ . The graph  $G'_2$  is claw-free.

Case 2.2.2.1.  $\delta(G'_2) \leq 1$ .

The fact that no vertex in  $N(x) - \{v_1, v_2\}$  has a degree two neighbor in  $V(G) - N[x]$ , implies that there is a vertex  $v \in V(G) - N[x]$ , adjacent to both  $w$  and  $w'$ , that has degree three. Hence,  $N(w) = \{v, w', x\}$  and  $N(w') = \{v, w, x\}$ . The graph  $G' = G - wv$  is claw-free, connected and has minimum degree at least two. By Observation 11, we get a contradiction.

Case 2.2.2.2.  $\delta(G'_2) \geq 2$ .

If  $G'_2 \notin \mathcal{K}$  then  $G'_2$  has a *TRDS*  $S_2$  of cardinality at most  $\frac{4(n-5)}{7}$ . If  $w$  ( $w'$  respectively) has a neighbor in  $V(G'_2) - S_2$ , then  $\{x, w'\} \cup S_2$  ( $\{x, w\} \cup S_2$  respectively) is a *TRDS* of  $G$ . If  $w$  (and  $w'$ ) has a neighbor in  $S_2$  then  $\{v_1, v_2\} \cup S_2$  is a *TRDS* of  $G$ . Hence  $\gamma_{\text{tr}}(G) \leq 2 + |S_2| < \frac{4n}{7} - \frac{20}{7} + \frac{14}{7} < \frac{4n}{7}$ , which is a contradiction. We may assume that  $G'_2$  has a component  $\mathcal{U}$  that is isomorphic to a graph in  $\mathcal{K}$ . Without loss of generality,  $u_1$  is adjacent to say  $w$ . Furthermore, the fact that  $\deg(w) = 3$  implies that  $N(w) = \{x, w', u_1\}$ .

If  $\mathcal{U} \in \mathcal{K} - \{C_3\}$ , then the first part of Observation 8 implies that  $w$  is adjacent to  $u_2$ . Hence,  $\deg(w) \geq 4$  which is a contradiction.

Suppose that  $\mathcal{U} = C_3$ . If  $V(G'_2) - V(\mathcal{U}) = \emptyset$ , then  $n(G) = 8$ . The set  $\{x, w, w', u_1\}$  is a *TRDS* of  $G$  and so  $\gamma_{\text{tr}}(G) \leq 4 < \frac{4.8}{7} = \frac{4n}{7}$ , which is a contradiction. If  $V(G'_2) - V(\mathcal{U}) \neq \emptyset$  then  $w'$  is adjacent to a vertex  $v \in V(G'_2) - V(\mathcal{U})$ . We let  $G'_1 = \langle V(\mathcal{U}) \cup \{w\} \rangle$  and  $G'_2 = G - G'_1$ . The graph  $G'_2$  is claw-free, has degree at least two and  $G'_2 \notin \mathcal{K}$ . By Observation 7, we get a contradiction.

We may assume that  $\mathcal{U} = \mathcal{B}$ . The second part of Observation 8 implies that  $w$  is adjacent to say  $u_1$ . If  $V(G'_2) - V(\mathcal{U}) = \emptyset$  then  $n(G) = 10$ . The set  $\{x, w', u_3, u_2\}$  is a *TRDS* of  $G$  and so  $\gamma_{\text{tr}}(G) \leq 4 < \frac{4.10}{7} = \frac{4n}{7}$ , which is a contradiction. If  $V(G'_2) - V(\mathcal{U}) \neq \emptyset$  then  $w'$  is adjacent to a vertex  $v \in V(G'_2) - V(\mathcal{U})$ . We let  $G'_1 = \langle V(\mathcal{U}) \cup \{w\} \rangle$  and  $G'_2 = G - G'_1$ . The graph  $G'_2$  is claw-free, has degree at least two and  $G'_2 \notin \mathcal{K}$ . This contradicts the second part of Observation 10.  $\square$

**Proof of Lemma 17.** Consider the 2-path  $P : x, v_1, y$  and suppose, to the contrary, that  $\mathcal{H}(P) \neq \emptyset$ . Let  $w \in \mathcal{H}(P)$ . Define  $G_1 = \langle V(P) \cup \{w\} \rangle$  and  $G_2 = G - G_1$ . Since the path  $P' : w, y, v_1, x$  is such that  $x$  and  $w$  have degree at least two and  $\deg(v_1) = 2$ , we may apply Lemma 14 and deduce that  $\delta(G_2) \leq 1$ . Let  $v$  be a vertex such that  $\deg_{G_2}(v) \leq 1$ .

Case 1:  $\deg_{G_2}(v) = 0$ .

Since  $\deg(v) \geq 2$ , we must have that  $v$  is adjacent to at least two vertices in the set  $\{x, y, w\}$ . It follows, without loss of generality, that  $v$  is adjacent to  $x$ . Since  $\langle N[x] - \{v_1, y\} \rangle$  is complete, we must have that  $\{w, x\} \subseteq N(v) \subseteq \{x, y, w\}$  and  $\{v, w, v_1\} \subseteq N(x) \subseteq \{v, w, v_1, y\}$ . We form the graph  $G'$  by removing, from  $G$ , the edge  $xw$ . Clearly,  $G'$  is connected and  $\delta(G') \geq 2$ . If  $G'$  is claw-free then, by [Observation 11](#), we are done. So we may assume that  $G'$  has a claw.

Let  $\{w_4, w_3, w_2, w_1\}$  be a claw of  $G'$ . We may also assume that  $w_2 = x$  and  $w_3 = w$ . Clearly,  $w_1 \in \{v, y\}$ . If  $w_1 = v$  then  $w_4 = y$ . But then  $w_4$  is adjacent to  $w_3$ , which is a contradiction. Hence,  $w_1 = y$ . If  $w_4 \in \{v_1, v\}$  then  $w_4$  is adjacent to  $w_2$  which is impossible. Hence,  $w_4 \in V(G) - V(P) - \{v, w\}$ . Since  $\langle N[y] - \{v_1, x\} \rangle$  is complete, we must have that  $w_4$  is adjacent to  $w_3$  which is a contradiction.

We may assume that no vertex of  $G_2$  has degree 0.

Case 2:  $\deg_{G_2}(v) = 1$ .

Case 2.1. The vertex  $v$  is adjacent to  $x$ .

Since  $\langle N[x] - \{v_1, y\} \rangle$  is a clique, we have that  $v$  is adjacent to  $w$ . Furthermore,  $v$  has exactly one neighbor  $z$  in  $V(G) - V(P) - \{w, v\}$ . Hence,  $\{x, w, z\} \subseteq N(v) \subseteq \{x, y, w, z\}$  and  $\deg(w) \geq 3$ .

If  $\deg(w) = 3$  then  $N(w) = \{x, y, v\}$  and, since  $\langle N[x] - \{v_1, y\} \rangle$  and  $\langle N[y] - \{v_1, x\} \rangle$  are complete, we have that  $\{v, w, v_1\} \subseteq N(x) \subseteq \{v, w, y, v_1\}$  and  $\{v_1, w\} \subseteq N(y) \subseteq \{v, w, x, v_1\}$ . The fact that  $\deg(y) \geq 3$ , implies that either  $vy$  or  $xy$  exists. We form the graph  $G'$  by removing, from  $G$ , the edges  $vy$  (if it exists) and  $xy$  (if it exists). If  $G'$  has a claw then, since  $\langle V(P) \cup \{v, w\} \rangle_{G'}$  and  $\langle V(G) - V(P) - \{w\} \rangle_{G'}$  are claw-free in  $G'$ , the center of this claw must be  $v$ . But then  $G$  has a claw which is a contradiction. Hence,  $G'$  is claw-free. Furthermore,  $\delta(G') \geq 2$  and  $G' \notin \mathcal{K}$ . The graph  $G'$  has a TRDS  $S$  of cardinality at most  $\frac{4n}{7}$ . Hence,  $\gamma_{tr}(G) \leq |S| \leq \frac{4n}{7}$  which is a contradiction. We may therefore assume that  $\deg(w) \geq 4$ .

Let  $G'_1 = \langle \{x, y, v_1, v\} \rangle$  and  $G'_2 = G - G'_1$ . From [Lemma 14](#) it follows that  $\delta(G'_2) \leq 1$ . Let  $v'$  be a vertex of  $G'_2$  that has degree at most one. If  $\deg_{G'_2}(v') = 0$  then, since  $\deg(w) \geq 4$ , we have that  $v' \neq w$  and  $v'$  is adjacent to two vertices in the set  $\{x, y, v\}$ . Since both  $\langle N[x] - \{v_1, y\} \rangle$  and  $\langle N[y] - \{v_1, x\} \rangle$  are complete, we have that  $v'$  is adjacent to  $w$ . But then  $\deg_{G'_2}(v') \geq 1$ , a contradiction. It follows that  $\deg_{G'_2}(v') = 1$ .

We claim that  $v' \in \{z, w\}$ . Suppose, to the contrary, that  $v' \notin \{z, w\}$ . Note that  $v'$  is adjacent to either  $x$  or  $y$ . If  $v'$  is adjacent to  $x$  then, since  $\langle N[x] - \{v_1, y\} \rangle$  is complete and  $\{x, w, z\} \subseteq N(v) \subseteq \{x, y, w, z\}$ , we have that  $v' \in \{w, z\}$  which is a contradiction. Hence,  $v'$  is adjacent to  $y$  and not to  $x$ . The fact that  $\langle N[y] - \{v_1, x\} \rangle$  is complete implies that  $N(v') = \{y, w\}$  and so  $\deg_{G_2}(v') = 0$ , which is a contradiction. Hence,  $v' \in \{z, w\}$ .

Case 2.1.1. The vertex  $z$  is adjacent to  $x$ .

If  $N(x) \cap (V(G) - V(P) - \{w, v, z\})$  has a vertex  $u$ , then, since  $\langle N[x] - \{v_1, y\} \rangle$  is complete, we will have that  $u \in N(v)$ . Hence,  $\deg_{G_2}(v) \geq 2$  which is a contradiction. It immediately follows that  $\{v_1, w, v, z\} \subseteq N(x) \subseteq \{v_1, w, v, z, y\}$ . Since  $\langle N[x] - \{v_1, y\} \rangle$  is complete, we have that  $w$  is adjacent to  $z$ .

Case 2.1.1.1.  $v' = z$ .

Note that  $\{w, v, x\} \subseteq N(z) \subseteq \{w, v, x, y\}$ . If  $\deg_{G'_2}(w) = 1$  then  $N(w) = \{x, y, z, v\}$  and if  $N(y) \cap (V(G) - V(P) - \{w, v, z\})$  has a vertex  $y'$  then, since  $\langle N[y] - \{v_1, x\} \rangle$  is complete, we will have that  $y' \in N(w)$  which is a contradiction. Hence,  $N(y) \subseteq \{x, w, v, z, v_1\}$  and so  $n(G) = 6$ . The set  $\{x, v_1, y\}$  is a TRDS of  $G$  and so  $\gamma_{tr}(G) \leq 3 < \frac{4.6}{7} = \frac{4n}{7}$ , a contradiction. It follows that  $\deg_{G'_2}(w) \geq 2$  and so  $w$  has a neighbor in  $V(G) - V(P) - \{v, z, w\}$ . We let  $G''_1 = \langle \{x, v_1, z, v\} \rangle$  and  $G''_2 = G - G''_1$ . Note that  $G''_2$  is connected.

If  $\delta(G''_2) = 1$  then the fact that  $w$  has a neighbor in  $V(G) - V(P) - \{v, z, w\}$ , implies that  $\deg_{G''_2}(y) = 1$  and so  $\{w, v_1\} \subseteq N(y) \subseteq \{w, v, z, v_1, x\}$ . We form the graph  $G'$  by first removing, from  $G$ , the edges between  $w$  and the vertices  $v, z$  and  $x$  and then removing the possible edges between  $y$  and the vertices  $v, z$  and  $x$ . If  $G'$  has a claw then, since  $\langle V(P) \cup \{z, v, w\} \rangle_{G'}$  and  $\langle V(G) - V(P) - \{z, v\} \rangle_{G'}$  are claw-free in  $G'$ , the center of this claw must be  $w$ . But then  $G$  has a claw, which is a contradiction. The graph  $G'$  is therefore claw-free and  $\delta(G') \geq 2$ . Furthermore,  $G' \notin \mathcal{K}$  and so  $G'$  has a TRDS  $S$  of cardinality at most  $\frac{4n}{7}$ . It follows that  $\gamma_{tr}(G) \leq |S| \leq \frac{4n}{7}$ , which is a contradiction.

Hence,  $\delta(G''_2) \geq 2$ . If  $G''_2 \notin \mathcal{K}$  then, by [Observation 7](#), we are done. Hence,  $G''_2 \in \mathcal{K}$ .

If  $G''_2 \in \mathcal{K} - \{C_3, \mathcal{B}\}$  then suppose, without loss of generality, that  $u_1 = y$  and  $u_2 = w$ . The fact that  $\langle N[y] - \{v_1, x\} \rangle$  is complete will imply that  $u_2$  is adjacent to  $u_i$ , a contradiction.

If  $G''_2 = C_3$  then  $n(G) = 7$ . The set  $\{x, v_1, y\}$  is a TRDS of  $G$ . Hence,  $\gamma_{tr}(G) \leq 3 < \frac{4.7}{7} = \frac{4n}{7}$  which is a contradiction.

If  $G''_2 = \mathcal{B}$  then  $n(G) = 9$ . The set  $\{x, v_1, u_1, u_2, u_3\}$  is a TRDS of  $G$ . Hence,  $\gamma_{tr}(G) \leq 5 < \frac{4.9}{7} = \frac{4n}{7}$  which is a contradiction.

Case 2.1.1.2.  $v' = w$ .

Clearly,  $N(w) = \{x, v, z, y\}$ . The fact that  $\langle N[y] - \{v_1, x\} \rangle$  is complete must imply that  $\{w, v_1\} \subseteq N(y) \subseteq \{x, v_1, z, v, w\}$ . We form the graph  $G'$  by first removing, from  $G$ , the possible edges between  $y$  and the vertices  $v, z$  and  $x$  and then removing the edges  $xw$  and  $wz$ . If  $G'$  has a claw then, since  $\langle V(P) \cup \{z, v, w\} \rangle_{G'}$  and  $\langle V(G) - V(P) - \{v, w\} \rangle_{G'}$  are claw-free in  $G'$ , the center of this claw must be  $z$ . But then  $G$  will contain a claw, a contradiction. Hence, the graph  $G'$  is claw-free and  $\delta(G') \geq 2$ . Furthermore,  $G' \notin \mathcal{K}$  and so  $G'$  has a TRDS  $S$  of cardinality at most  $\frac{4n}{7}$ . It follows that  $\gamma_{tr}(G) \leq |S| \leq \frac{4n}{7}$ , which is a contradiction.

Case 2.1.2. The vertex  $z$  is not adjacent to  $x$ .

Since  $\langle N[x] - \{v_1, y\} \rangle$  is complete and  $\{x, w, z\} \subseteq N(v) \subseteq \{x, y, w, z\}$ , we have that  $\{v_1, v, w\} \subseteq N(x) \subseteq \{v_1, v, y, w\}$ .



**Claim 1.**  $z$  is adjacent to  $w$ .

**Proof.** We form the graph  $G'$  by removing, from  $G$ , the edges  $vw$ ,  $xw$  and  $vy$  (if it exists). Since  $\deg(w) \geq 4$ , we have that  $\delta(G') \geq 2$ . Suppose first that  $G'$  is claw-free. If  $G' \notin \mathcal{K}$  then we have that  $G'$  has a  $TRDS$   $S$  of cardinality at most  $\frac{4n}{7}$ . It follows that  $\gamma_{tr}(G) \leq |S| \leq \frac{4n}{7}$ , a contradiction. We may therefore assume that  $G' \in \mathcal{K} - \{C_3\}$ .

If  $G' = \mathcal{B}$  then it is easy to verify that joining at least one pair of non-adjacent vertices of  $\mathcal{B}$  will result in a graph that has a  $TRDS$  of cardinality at most  $\frac{4n}{7}$ . Hence,  $G$  has a  $TRDS$  of cardinality at most  $\frac{4n}{7}$  which is a contradiction.

If  $G' \in \mathcal{K} - \{C_3, \mathcal{B}\}$  then  $G$  has a Hamiltonian cycle. Recall that  $G$  contains a triangle. If at least one edge is added to a graph in  $\mathcal{K} - \{C_3\}$  such that the resulting graph contains a triangle, then it is easy to verify that this resulting graph has a  $TRDS$  of cardinality at most  $\frac{4n}{7}$ . Hence,  $G$  has a  $TRDS$  of cardinality at most  $\frac{4n}{7}$  which is a contradiction.

We may assume that  $G'$  is not claw-free. Let  $\langle \{w_4, w_3, w_2, w_1\} \rangle$  be a claw in  $G'$ . Suppose first, without loss of generality, that  $w_2 = x$  and  $w_3 = w$ . Since  $\{v_1, v, w\} \subseteq N(x) \subseteq \{v_1, v, y, w\}$  and  $\{x, w, z\} \subseteq N(v) \subseteq \{x, y, w, z\}$ , we have that  $w_1 = y$  and so  $y$  and  $x$  are adjacent. If  $w_4 = v_1$  then  $w_4$  and  $w_2$  are adjacent, which is a contradiction. Hence,  $w_4 \in V(G) - V(P) - \{w, v\}$ . Since  $\langle N[y] - \{v_1, x\} \rangle$  is complete, we must have that  $w_4$  is adjacent to  $w_3$  which is a contradiction. We may assume that  $w_2 = v$  and that  $w_3 \in \{y, w\}$ .

If  $w_3 = w$  then the fact that  $\{x, w, z\} \subseteq N(v) \subseteq \{x, y, w, z\}$ , implies that  $w_1 = z$  and so  $z$  is adjacent to  $w$ . Hence,  $w_2 = v$  and  $w_3 = y$ . Clearly,  $w_1 \in \{x, z\}$ . Suppose first that  $w_1 = x$ . Then  $x$  is adjacent to  $y$ . The fact that  $\{v_1, v, w\} \subseteq N(x) \subseteq \{v_1, v, y, w\}$ , implies that  $w_4 = v_1$ . But then  $w_4$  is adjacent to  $w_3$ , a contradiction. Hence,  $w_1 = z$  and so  $z$  must be adjacent to  $y$ . Since  $\langle N[y] - \{v_1, x\} \rangle$  is complete, we have that  $z$  is adjacent to  $w$ .  $\square$

**Case 2.1.2.1.**  $v' = z$ .

Clearly,  $\deg(z) = 2$  or  $\deg(z) = 3$  and  $N(z) = \{v, w, y\}$ . If  $w$  has a neighbor  $w'$  in  $V(G) - V(P) - \{w, v, z\}$ , then  $\langle \{z, x, w', w\} \rangle$  will induce a claw which is a contradiction. Hence  $N(w) = \{v, z, x, y\}$  and since  $\langle N[y] - \{v_1, x\} \rangle$  is complete, we have that  $\{v_1, w\} \subset N(y) \subseteq \{v, z, v_1, x, w\}$ . Hence,  $n(G) = 6$  and the set  $\{x, w\}$  is a  $TRDS$  of  $G$ . Hence,  $\gamma_{tr}(G) \leq 2 < \frac{4.6}{7} = \frac{4n}{7}$  which is a contradiction.

**Case 2.1.2.2.**  $v' = w$ .

Clearly,  $N(w) = \{x, y, v, z\}$ . Let  $G' = G - xw$ . It follows that  $\delta(G') \geq 2$  and if  $G'$  is claw-free then, by [Observation 11](#), we are done. Hence,  $G'$  is not claw-free. Let  $\langle \{w_4, w_3, w_2, w_1\} \rangle$  be a claw in  $G'$ . We may assume, without loss of generality, that  $x = w_2$  and  $w = w_3$ . Since  $z$  is not adjacent to  $x$  and  $\{v_1, v, w\} \subseteq N(x) \subseteq \{v_1, v, y, w\}$ , we have that  $w_1 \in \{y, v\}$ .

Suppose that  $w_1 = v$ . If  $w_4 = y$  then  $w_4$  is adjacent to  $w_3$ , a contradiction. Hence,  $w_4 = z$ . But then  $w_4$  is adjacent to  $w_3$ , a contradiction. Hence,  $w_1 = y$  and  $y$  is adjacent to  $x$ . If  $w_4 = v_1$  then  $w_2$  is adjacent to  $w_4$ , a contradiction. Hence,  $w_4 \in V(G) - V(P) - \{w\}$ . Since  $\langle N[y] - \{v_1, x\} \rangle$  is complete, we must have that  $w_4$  is adjacent to  $w_3$  which is a contradiction.

**Case 2.2** The vertex  $v$  is not adjacent to  $x$ .

We may assume, by symmetry, that any vertex of  $G_2$  that has degree at most one, is adjacent to neither  $x$  nor  $y$ . Hence,  $v$  is not adjacent to  $y$ . Clearly,  $\deg(v) = 2$ . Let  $z \in N(v) - \{w\}$ . Since  $G$  has no 2-paths of length greater than two we have that  $\deg(z) \geq 3$ , or  $\deg(z) = 2$  and  $N(z) = \{w, v\}$ . If  $N(x) \cap (V(G) - V(P)) - \{w\} = N(y) \cap (V(G) - V(P)) - \{w\} = \emptyset$  then it follows that  $N(x) = \{v_1, y, w\}$  and  $N(y) = \{v_1, x, w\}$ . The graph  $G' = G - wy$  is claw-free and  $\delta(G') \geq 2$ . By [Observation 11](#) we are done. Hence,  $N(x) \cap (V(G) - V(P)) - \{w\} \neq \emptyset$  or  $N(y) \cap (V(G) - V(P)) - \{w\} \neq \emptyset$ .

**Claim 2.** If  $u$  is adjacent to either  $x$  or  $y$  and  $u \notin \{z, w\}$ , then  $u \in \mathcal{H}(P)$ .

**Proof.** Suppose, without loss of generality, that  $u$  is adjacent to  $x$  and  $u \notin \{z, w\}$ . Since  $\langle N[x] - \{v_1, y\} \rangle$  is complete, we have that  $u$  is adjacent to  $w$ . If  $z = y$  ( $z = x$  respectively) then  $v$  is adjacent to  $y$  ( $x$  respectively), a contradiction. The fact that  $\langle \{v, y, u, w\} \rangle$  does not induce a claw, implies that  $u$  is also adjacent to  $y$ . Hence,  $u \in \mathcal{H}(P)$  and our claim is verified.  $\square$

**Case 2.2.1.**  $z$  is adjacent to either  $x$  or  $y$ .

Without loss of generality, suppose that  $z$  is adjacent to  $x$ . Since  $\langle N[x] - \{v_1, y\} \rangle$  is complete, we have that  $z$  is adjacent to  $w$ . Note that if  $z$  ( $w$  respectively) has a neighbor  $u$  in  $V(G) - N[x] - N[y] - \{v\}$ , then  $\langle \{x, v, u, z\} \rangle$  ( $\langle \{x, v, u, w\} \rangle$  respectively) induces a claw which is a contradiction. Hence,  $N(z) \cap (V(G) - N[x] - N[y]) = N(w) \cap (V(G) - N[x] - N[y]) = \{v\}$ . Let  $G'_1 = \langle \{v, z, x, v_1\} \rangle$  and  $G'_2 = G - G'_1$ . Since  $\deg(v_1) = \deg(v) = 2$  we have, by [Lemma 13](#), that  $\delta(G'_2) \leq 1$ .

Let  $v'$  be a vertex of  $G'_2$  that has degree at most one. We claim that  $v' \in \{y, w\}$ . Suppose, to the contrary, that  $v' \notin \{y, w\}$ . If  $v'$  is adjacent to  $x$  then, since  $\langle N(x) - \{v_1, y\} \rangle$  is complete, we have that  $v'$  is adjacent to  $w$  and  $z$ . Since  $v' \notin \{z, w\}$  we have, by Claim 2, that  $v'$  is adjacent to  $y$ . Hence,  $\deg_{G'_2}(v') \geq 2$  which is a contradiction. We may assume that  $v'$  is not adjacent to  $x$ . Hence,  $v'$  is adjacent to  $z$ . Since  $N(z) \cap (V(G) - N[x] - N[y]) = \{v\}$ , we can conclude that  $v' \in N(y)$ . Claim 2 implies that  $v'$  must be adjacent to  $x$  which is impossible. We can conclude that  $v' \in \{y, w\}$ .

If  $v' = y$  then  $\{v_1, w\} \subset N(y) \subseteq \{v_1, x, z, w\}$ . If  $x$  is adjacent to a vertex  $u \in V(G) - V(P) - \{z, w, v\}$ , then since  $u \notin \{z, w\}$  we have, by Claim 2, that  $u$  is adjacent to  $y$  and so this contradicts the fact that  $N(y) \subseteq \{v_1, x, z, w\}$ . Hence,  $\{v_1, z, w\} \subseteq N(x) \subseteq \{v_1, y, z, w\}$  and  $\{w, v, x\} \subseteq N(z) \subseteq \{y, x, v, w\}$ . Note that  $z$  is adjacent to  $x$  and  $\deg_{G'_2}(z) = 1$ . This contradicts our earlier assumption.

If  $v' = w$  then  $N(w) = \{x, y, v, z\}$ . Since  $\langle N[x] - \{v_1, y\} \rangle$  and  $\langle N[y] - \{v_1, x\} \rangle$  are complete, we must have that  $\{v_1, w\} \subset N(y) \subseteq \{v_1, x, z, w\}$  and  $\{v_1, w, z\} \subseteq N(x) \subseteq \{v_1, y, z, w\}$ . We have, once more, that  $z$  is adjacent to  $x$  and  $\deg_{G'_2}(z) = 1$ , contradicting our earlier assumption.



Case 2.2.2.  $z$  is adjacent to neither  $x$  nor  $y$ .

By Claim 2,  $y$  ( $x$  respectively) is adjacent to every vertex of  $N(x)$  ( $N(y)$  respectively). Hence,  $N(x) \cap (V(G) - V(P)) = N(y) \cap (V(G) - V(P))$ . Furthermore, since  $\langle N(x) - \{v_1, y\} \rangle$  and  $\langle N(y) - \{v_1, x\} \rangle$  are complete, we have that  $\langle N(x) \cup N(y) - \{v_1, x, y\} \rangle$  is complete. Since  $N(x) \cap (V(G) - V(P)) - \{w\} \neq \emptyset$  or  $N(y) \cap (V(G) - V(P)) - \{w\} \neq \emptyset$  we let, without loss of generality,  $u$  be a vertex in  $N(x) \cap (V(G) - V(P) - \{w\})$ . The vertex  $u$  is adjacent to  $w$  and  $y$ . Note that if  $w$  is adjacent to a vertex  $w'$  in  $V(G) - N[x] - N[y] - \{v, z\}$ , then  $\langle \{x, w', v, w\} \rangle$  induces a claw which is a contradiction. Hence,  $N(w) \cap (V(G) - N[x] - N[y]) \subseteq \{z, v\}$ . If  $x$  and  $y$  are not adjacent then  $\langle \{x, y, v, w\} \rangle$  induces a claw which is a contradiction. Hence,  $\langle N(w) - \{z, v\} \rangle$  is complete.

Case 2.2.2.1.  $\deg(z) = 2$ .

Since  $G$  has no 2-paths of length greater than two, we have that  $N(z) = \{w, v\}$ . Due to the fact that  $\langle N(x) \cup N(y) - \{v_1, x, y\} \rangle$  is complete and  $N(w) \cap (V(G) - N[x] - N[y]) \subseteq \{z, v\}$ , we can deduce that  $N(w) = N[x] \cup N[y] \cup \{z, v\} - \{v_1, w\}$ . Let  $G'_1 = \langle V(P) \cup \{w, z, v\} \rangle$  and note that  $G'_1$  induces a necklace where the vertices of the attachment are the vertices of  $V(P)$ . Let  $G'_2 = G - G'_1$ . If  $\delta(G'_2) \leq 1$  then there will be a vertex  $v'$ , of  $G'_2$ , of degree at most one. Furthermore, since  $\deg(v') \geq 2$  we will have that  $v'$  is adjacent to at least one vertex in  $\{x, y\}$ . Hence,  $\deg_{G'_2}(v') \leq 1$  and this contradicts our earlier assumption. Hence,  $\delta(G'_2) \geq 2$ .

We first claim that  $G'_2$  is connected. Suppose, to the contrary, that  $G'_2$  has two components  $\mathcal{U}$  and  $\mathcal{U}'$ . Since  $G$  is connected and  $N(w) = N[x] \cup N[y] \cup \{z, v\} - \{v_1, w\}$ , we have that  $\mathcal{U}$  ( $\mathcal{U}'$  respectively) has a vertex  $u'$  ( $u''$  respectively) in  $\langle N(x) \cup N(y) - \{v_1, w, x, y\} \rangle$ . Since  $\langle N(x) \cup N(y) - \{v_1, x, y\} \rangle$  is complete we have that  $\langle N(x) \cup N(y) - \{v_1, w, x, y\} \rangle$  is complete and so  $u'$  is adjacent to  $u''$  which is a contradiction. Hence,  $G'_2$  is connected.

If  $G'_2 \notin \mathcal{K}$  then, by [Observation 6](#), we are done. Hence,  $G'_2 \in \mathcal{K}$ .

If  $G'_2 = C_3$ , then  $n(G) = 9$ . Without loss of generality,  $u_1$  is adjacent to  $x$  or  $w$  and so we have that the set  $\{x, w, u_1\}$  is a TRDS of  $G$  and so  $\gamma_{\text{tr}}(G) \leq 3 < \frac{4.9}{7} = \frac{4n}{7}$ , which is a contradiction.

If  $G'_2 = \mathcal{B}$  then  $n(G) = 11$ . The set  $\{x, w, u_1, u_2, u_3\}$  is a TRDS of  $G$  and so  $\gamma_{\text{tr}}(G) \leq 5 < \frac{4.11}{7} = \frac{4n}{7}$ , which is a contradiction.

Hence,  $G'_2 \in \mathcal{K} - \{C_3, \mathcal{B}\}$ . Note that, without loss of generality,  $u_1$  is adjacent to some vertex in  $\{x, y, w\}$ . Since  $N(w) = N[x] \cup N[y] \cup \{z, v\} - \{w, v_1\}$  and  $N(x) \cap (V(G) - V(P)) = N(y) \cap (V(G) - V(P))$ , we have that  $x, y$  and  $w$  are adjacent to say  $u_1$ . Hence, we can say that any vertex of  $G'_2$  that is adjacent to a vertex in  $\{x, y, w\}$  is adjacent to  $x, y$  and  $w$ . Hence, [Observation 8](#) implies, without loss of generality, that  $u_2$  is adjacent to  $x, y$  and  $w$ . Furthermore, if  $v'' \in \{x, y, w\}$  is adjacent to a vertex  $u_j$ , for  $3 \leq j \leq i-1$ , then  $u_j \in N(x)$  and since  $\langle N[x] - \{v_1, y\} \rangle$  is complete, we have that  $u_j$  and  $u_1$  are adjacent which is a contradiction. Similarly, if  $v''$  is adjacent to  $u_i$  then  $u_i \in N(x)$  and so  $u_2$  and  $u_i$  will be adjacent which is a contradiction. Hence,  $P' : u_2, \dots, u_i, u_1$  is a 2-path with length greater than three which, by [Lemma 12](#), is a contradiction.

Case 2.2.2.2. The vertex  $\deg(z) \geq 3$ .

Let  $G'_1 = \langle \{v, w, x, v_1\} \rangle$  and  $G'_2 = G - G'_1$ . If  $\delta(G'_2) \geq 2$  then, by [Lemma 13](#), we are done. We may assume that  $\delta(G'_2) \leq 1$ . Let  $v'$  be the vertex of  $G'_2$  that has degree at most one. We claim that  $v' \in \{y, z\}$ . Suppose, to the contrary, that  $v' \notin \{y, z\}$ . Since  $\langle N[x] - \{v_1, y\} \rangle$  is complete and  $N(w) \cap (V(G) - N[x] - N[y]) \subseteq \{z, v\}$ , we have that  $N(v') = \{w, x, y\}$ . But then  $\deg_{G'_2}(v') = 0$ , a contradiction. Hence,  $v' \in \{y, z\}$ .

Suppose that  $v' = y$ . Clearly,  $N(y) = \{v_1, u, w, x\}$ . Since  $N(x) \cap (V(G) - V(P)) = N(y) \cap (V(G) - V(P))$ , we can deduce that  $N(x) = \{v_1, u, w, y\}$ . Since  $N(w) \cap (V(G) - N[x] - N[y]) \subseteq \{z, v\}$ , we have that  $\{u, x, y, v\} \subseteq N(w) \subseteq \{u, x, y, v, z\}$ . We form  $G'$  by removing, from  $G$ , the edges  $wy$  and  $uy$ . The graph  $G'$  is such that  $\delta(G') \geq 2$  and  $G' \notin \mathcal{K}$ . If  $G'$  is claw-free then  $G'$  has a TRDS  $S$  of cardinality at most  $\frac{4n}{7}$ . It follows that  $\gamma_{\text{tr}}(G) \leq |S| \leq \frac{4n}{7}$ , a contradiction. Hence,  $G'$  has a claw  $\langle \{w_4, w_3, w_2, w_1\} \rangle$ . Clearly,  $w_2 = y$  and  $w_3 \in \{w, u\}$ .

If  $w_3 = u$  then since  $N(y) = \{v_1, u, w, x\}$ , we have that  $w_1 = x$ . Since  $N(x) = \{v_1, u, w, y\}$  we have that  $w_4 \in \{v_1, w\}$ . If  $w_4 = v_1$  then  $w_4$  is adjacent to  $w_2$ , a contradiction. If  $w_4 = w$  then  $w_4$  is adjacent to  $w_3$ , a contradiction.

Hence,  $w_3 = w$ . Since  $N(y) = \{v_1, u, w, x\}$  and  $N(x) = \{v_1, u, w, y\}$ , we have that  $w_1 = x$  and  $w_4 \in \{v_1, u\}$ . If  $w_4 = v_1$  then  $w_4$  is adjacent to  $w_2$ , a contradiction. If  $w_4 = u$  then  $w_4$  is adjacent to  $w_3$ , a contradiction.

Hence  $v' = z$ ,  $\deg(z) = 3$  and  $z$  is adjacent to  $v$  and  $w$ . The fact that  $N(w) \cap (V(G) - N[x] - N[y]) \subseteq \{z, v\}$ , implies that  $N(w) = N[x] \cup N[y] \cup \{z, v\} - \{w, v_1\}$ .

Case 2.2.2.2.1.  $z$  is adjacent to a vertex in  $N(x) \cup N(y) - \{w, v_1, x, y\}$ .

Without loss of generality,  $z$  is adjacent to say  $u$ . Hence,  $N(z) = \{w, u, v\}$ . Let  $G' = G - uz$ . Clearly,  $G'$  is connected and  $\delta(G') \geq 2$ . If  $G'$  is claw-free then, by [Observation 11](#), we are done. Hence,  $G'$  has a claw  $\langle \{w_4, w_3, w_2, w_1\} \rangle$ . Furthermore,  $w_2 = u$  and  $w_3 = z$ . Since  $N(z) = \{w, u, v\}$ , we must have that  $w_1 = w$ . Furthermore,  $w_4 \in N[x] \cup N[y] \cup \{v\} - \{w, v_1\}$ . If  $w_4 = v$ , then  $w_4$  and  $w_3$  will be adjacent which is a contradiction. Hence,  $w_4 \in N[x] \cup N[y] - \{w, v_1\}$ . The fact that  $\langle N(w) - \{z, v\} \rangle$  is complete, implies that  $w_4$  is adjacent to  $w_2$  which is a contradiction.

Case 2.2.2.2.2.  $z$  is not adjacent to any vertex in  $N(x) \cup N(y) - \{v_1, w, x, y\}$ .

We may assume that  $z$  has a neighbor  $z'$  in  $V(G) - N[x] - N[y] - \{v, z\}$ . Hence,  $N(z) = \{w, v, z'\}$ . Let  $G' = G - wz$ . Clearly,  $G'$  is connected and  $\delta(G') \geq 2$ . If  $G'$  is claw-free then, by [Observation 11](#), we are done. Hence,  $G'$  has a claw  $\langle \{w_4, w_3, w_2, w_1\} \rangle$ . Furthermore,  $w_2 = w$  and  $w_3 = z$ . Clearly,  $w_1 = z'$ . But since  $N(w) = N[x] \cup N[y] \cup \{z, v\} - \{w, v_1\}$ , we have that  $w_1$  cannot be adjacent to  $w_2$  which is impossible.  $\square$

**Proof of Lemma 18.** Suppose, to the contrary, that  $G$  has a 2-path  $P : x, v_1, y$ . We may assume, by [Lemma 17](#), that  $\mathcal{H}(P) = \emptyset$ . If  $x$  is adjacent to  $y$  then, since  $\mathcal{H}(P) = \emptyset$ , we have that the graph  $G' = G - xy$  is claw-free. Furthermore,  $G'$  is connected

and  $\delta(G') \geq 2$  and so, by [Observation 11](#), we are done. Thus,  $x$  is not adjacent to  $y$ . We may assume, henceforth, that for any 2-path of length two, the end vertices are not adjacent and the only vertex that is adjacent to both end vertices is the degree two vertex on the path. The proof of [Lemma 18](#) will follow from a series of claims.

**Claim 1.** The only degree two vertex adjacent to  $x$  ( $y$  respectively) is  $v_1$ .

**Proof.** Suppose, without loss of generality, that  $u$  is a degree two vertex in  $N(x) - \{v_1\}$ . Since  $\deg(x) \geq 3$ , there is a vertex  $v$  in  $N(x) - \{v_1, u\}$ . Furthermore, the fact that  $\langle N(x) - \{v_1\} \rangle$  is complete must imply that  $N(u) = \{v, x\}$  and  $N(x) = \{v, u, v_1\}$ . If  $\deg(v) \geq 3$  then the path  $P' : x, u, v$  is a 2-path of length two such that  $x$  is adjacent to  $v$ . This contradicts our earlier assumption. Hence,  $\deg(v) = 2$ . Define  $G_1 = \langle \{u, v, x, v_1\} \rangle$  and  $G_2 = G - G_1$ . Observe that  $\delta(G_2) \geq 2$ . This contradicts [Lemma 14](#).  $\square$

Let  $u$  be an arbitrary vertex in  $N(x) - \{v_1\}$ . Define  $G_1 = \langle \{y, u, x, v_1\} \rangle$  and  $G_2 = G - G_1$ . We have the following:

**Claim 2.** If no neighbor of  $y$  has degree one in  $G_2$ , then at least one of the following holds:

1. There is a vertex in  $N(x) - \{v_1, u\}$  that has degree one in  $G_2$ .
2. There are two adjacent degree two vertices in  $V(G) - N[x] - N[y]$ , both being adjacent to  $u$ .

**Proof.** Suppose that no neighbor of  $y$  has degree one in  $G_2$ . By [Lemma 14](#), we must have that  $\delta(G_2) \leq 1$ . Let  $v$  be a vertex of  $G_2$  that has degree at most one. If  $v \in N(x) - \{v_1, u\}$  then by Claim 1, and since  $\mathcal{H}(P) = \emptyset$ , we have that  $\deg_{G_2}(v) = 1$ . Hence, we are done.

It follows that  $v \in V(G) - N[x] - N[y]$  and  $\deg(v) = 2$ . Let  $z \in N(v) - \{u\}$ . If  $\deg(z) = 2$ , then, since  $G$  has no 2-paths of length greater than two, we have that  $z$  is adjacent to  $u$  and so we are done. Hence,  $\deg(z) \geq 3$ . Let  $G'_1 = \langle \{u, v, x, v_1\} \rangle$  and  $G'_2 = G - G'_1$ . By [Lemma 13](#) we have that  $\delta(G'_2) \leq 1$ . Let  $v'$  be a vertex of  $G'_2$  that has degree at most one. Since  $\langle N(y) - \{v_1\} \rangle$  is complete and  $\deg(y) \geq 3$ , we have that  $v' \in N(x) - \{v_1, u\}$  or  $v' \in V(G) - N[x] - N[y] - \{v\}$ .

**Case 1.**  $v' \in N(x) - \{v_1, u\}$ .

If  $v' \neq z$  then the fact that  $v'$  is not adjacent to  $y$  must imply that  $\deg_{G_2}(v') \leq 1$ . If  $\deg_{G_2}(v') = 0$  then  $\deg(v') = 2$ , which, by Claim 1, is a contradiction. Hence,  $\deg_{G_2}(v') = 1$  and so we are done. Hence,  $v' = z$ . Note that the path  $P' : u, v, v'$  is a 2-path of length two such that  $x \in \mathcal{H}(P')$ . This contradicts [Lemma 17](#).

**Case 2.**  $v' \in V(G) - N[x] - N[y] - \{v\}$ .

If  $v'$  is adjacent to  $v$  but not to  $u$ , then  $\deg(v') = 2$  and  $v' = z$  and so we contradict the fact that  $\deg(z) \geq 3$ . Hence,  $v'$  is adjacent to  $u$ . If  $v'$  is not adjacent to  $v$  then  $\langle \{x, v, v', u\} \rangle$  is a claw. Hence,  $v'$  is adjacent to  $v$  and so  $v' = z$ . The path  $P' : u, v, z$  is a 2-path of length two such that  $u$  and  $z$  are adjacent. This contradicts our earlier assumption.

This completes the proof of our claim.  $\square$

**Case 1.**  $\deg(x) \geq 4$ .

Let  $u$  be an arbitrary vertex of  $N(x) - \{v_1\}$ . Let  $v, w \in N(x) - \{v_1, u\}$ . Consider, once again, the graph  $G_2$ . By [Lemma 14](#) we have that  $\delta(G_2) \leq 1$ .

**Claim 3.** If  $G_2$  has no degree one vertex in  $N(y) - \{v_1\}$ , then  $G_2$  has no degree one vertex in  $N(x) - \{v_1, u\}$ .

**Proof.** Suppose that  $G_2$  has no degree one vertex in  $N(y) - \{v_1\}$ . Suppose, to the contrary, that say  $v$  has degree one in  $G_2$ . Since  $\langle N(x) - \{v_1\} \rangle$  is complete, we have that  $N(v) = \{w, x, u\}$  and  $N(x) = \{u, v, w, v_1\}$ . Define  $G'_1 = \langle \{v, y, x, v_1\} \rangle$  and  $G'_2 = G - G'_1$ . By [Lemma 14](#) we have that  $\delta(G'_2) \leq 1$ . Since  $v$  is not adjacent to any vertex in  $V(G) - N[x] - \{y\}$  we have, by Claim 1, that no neighbor of  $y$  has degree at most one in  $G'_2$ . Furthermore, Claim 2 implies, without loss of generality, that  $\deg_{G'_2}(w) = 1$  and so  $N(w) = \{v, u, x\}$ . Define the graph  $G' = G - xu$ . The graph  $G'$  has minimum degree at least two and is connected. If  $G'$  is claw-free then, by [Observation 11](#), we are done.

Hence,  $G'$  has a claw  $\langle \{w_4, w_3, w_2, w_1\} \rangle$ . Clearly,  $w_2 = x$  and  $w_3 = u$ . Furthermore, since  $N(x) = \{u, v, w, v_1\}$  we have that  $w_1 \in \{v, w\}$ . If  $w_1 = v$  ( $w_1 = w$  respectively) then since  $N(w) = \{v, u, x\}$  ( $N(v) = \{w, x, u\}$  respectively) we have that  $w_4 = w$  ( $w_4 = v$  respectively). In both cases  $w_4$  is adjacent to  $w_3$ , a contradiction.  $\square$

Let  $u'$  and  $v'$  be two vertices of  $N(y) - \{v_1\}$ . Let  $z$  be an arbitrary vertex in  $\{u, v, w\}$  and let  $z' \in \{u, v, w\} - \{z\}$ .

**Case 1.1.**  $\deg(y) = 3$  and neither  $u'$  nor  $v'$  is adjacent to a vertex in  $V(G) - N[x] - N[y]$ .

Let  $G'_1 = \langle \{u', v', y, v_1\} \rangle$  and  $G'_2 = G - G'_1$ . The graph  $G'_2$  has minimum degree at least two, is claw-free and is connected. Furthermore,  $G'_2 \notin \mathcal{K}$ . By [Observation 7](#) we are done.

**Case 1.2.**  $\deg(y) \geq 4$  or  $\deg(y) = 3$  and at least one vertex in  $\{u', v'\}$  is adjacent to a vertex in  $V(G) - N[x] - N[y]$ .

Suppose first that  $\deg(y) = 3$ . If the graph  $G - \langle \{z, y, x, v_1\} \rangle$  has no degree one vertex in  $N(y) - \{v_1\}$  for every  $z \in \{u, v, w\}$ , then neither  $G - \langle \{u, y, x, v_1\} \rangle$  nor  $G - \langle \{v, y, x, v_1\} \rangle$  has a degree one vertex in  $N(y) - \{v_1\}$ . If the graph  $G - \langle \{z, y, x, v_1\} \rangle$  has a degree one vertex in  $N(y) - \{v_1\}$  then, without loss of generality,  $u'$  has degree one in  $G - \langle \{z, y, x, v_1\} \rangle$  and so  $N(u') = \{z, y, v'\}$ . This implies that  $u'$  has no neighbor in  $V(G) - N[x] - N[y]$ . Hence,  $v'$  must have a neighbor in  $V(G) - N[x] - N[y]$  and  $\{y, u'\} \subset N(v')$ . It follows that for all  $z' \in \{u, v, w\} - \{z\}$ , we have that the graph  $G - \langle \{z', y, x, v_1\} \rangle$  has no degree one vertex in  $N(y) - \{v_1\}$ . Hence we may assume, without loss of generality, that neither  $G - \langle \{u, y, x, v_1\} \rangle$  nor  $G - \langle \{v, y, x, v_1\} \rangle$  has a degree one vertex in  $N(y) - \{v_1\}$ . If  $\deg(y) \geq 4$  then neither  $G - \langle \{u, y, x, v_1\} \rangle$  nor  $G - \langle \{v, y, x, v_1\} \rangle$  has a degree one vertex in  $N(y) - \{v_1\}$ .

By Claims 2 and 3, we have that  $u$  is adjacent to two adjacent degree two vertices  $u''$  and  $u'''$  in  $V(G) - N[x] - N[y]$ . If  $u$  is adjacent to some vertex  $w'$  in  $V(G) - N[x] - \{y\}$ , then  $\{w', x, u'', u\}$  is a claw which is a contradiction. Hence,  $N(u) = \{u'', u'''\} \cup N[x] - \{v_1, u\}$ . By the same argument,  $v$  is adjacent to two adjacent degree two vertices  $v''$  and  $v'''$  in  $V(G) - N[x] - N[y] - \{u'', u'''\}$ . Furthermore,  $N(v) = \{v'', v'''\} \cup N[x] - \{v, v_1\}$ .

Let  $G'_1 = \langle \{u, v, u'', u''', v'', v'''\} \rangle$  and  $G'_2 = G - G'_1$ . If  $\delta(G'_2) \geq 2$  then note that  $G'_2 \notin \mathcal{K}$ . Hence,  $G'_2$  has a TRDS  $S$  of cardinality at most  $\frac{4(n-6)}{7}$ . The set  $S \cup \{u, v\}$  is a TRDS of  $G$  and so  $\gamma_{tr}(G) \leq |S| + 2 \leq \frac{4n}{7} - \frac{24}{7} + \frac{14}{7} \leq \frac{4n}{7}$ , a contradiction. Thus,  $\delta(G'_2) \leq 1$ . Clearly,  $\deg_{G'_2}(w) \leq 1$  and so  $N(w) = \{v, x, u\}$ . Note that  $w$  has degree one in  $G - \langle \{u, y, x, v_1\} \rangle$  and, by Claim 1, we have that no neighbor of  $y$  has degree one in  $G - \langle \{u, y, x, v_1\} \rangle$ . This contradicts Claim 3.

Case 2.  $\deg(x) = 3$ .

By symmetry it follows that  $\deg(y) = 3$ . Let  $u, v \in N(x) - \{v_1\}$  and  $u', v' \in N(y) - \{v_1\}$ . Furthermore,  $N(x) = \{u, v, v_1\}$  and  $N(y) = \{u', v', v_1\}$ .

**Claim 4.** Every vertex in  $N(x) \cup N(y) - \{v_1\}$  has at least one neighbor in  $V(G) - N[x] - N[y]$ .

**Proof.** Suppose, to the contrary, that say  $u$  is adjacent to no vertex in  $V(G) - N[x] - N[y]$ . By Claim 1 and the fact that  $\langle N(x) - \{v_1\} \rangle$  is complete, we have that  $\{x, v\} \subset N(u) \subseteq \{v, x, v', u'\}$ . Suppose that  $u$  is adjacent to say  $z$ , where  $z \in \{u', v'\}$ . If  $z$  is adjacent to some vertex  $w$  in  $V(G) - N[x] - N[y]$ , then  $\{u, y, w, z\}$  is a claw which is a contradiction. Hence,  $z$  is adjacent to no vertex in  $V(G) - N[x] - N[y]$ .

Case 1.  $u$  is adjacent to  $u'$  and  $v'$ .

It immediately follows that neither  $u'$  nor  $v'$  is adjacent to any vertex in  $V(G) - N[x] - N[y]$ . Let  $G'_1 = \langle \{u', v', y, v_1\} \rangle$  and  $G'_2 = G - G'_1$ . The graph  $G'_2$  has minimum degree at least two, is claw-free and is connected. If  $G'_2 \notin \mathcal{K}$  then, by [Observation 7](#), we are done. It follows that  $G'_2 \in \mathcal{K}$ . Since  $x$  lies on a triangle, we have that  $G'_2 \in \{C_3, \mathcal{B}\}$ .

If  $G'_2 = C_3$  then  $n(G) = 7$  and the set  $\{x, y, v_1\}$  is a TRDS of  $G$  and so  $\gamma_{tr}(G) \leq 3 < \frac{4.7}{7} = \frac{4n}{7}$ , which is a contradiction.

If  $G'_2 = \mathcal{B}$  then  $n(G) = 9$  and the set  $\{u_1, u_2, u_3, y, v_1\}$  is a TRDS of  $G$  and so  $\gamma_{tr}(G) \leq 5 < \frac{4.9}{7} = \frac{4n}{7}$ , which is a contradiction.

Case 2.  $u$  is adjacent to say  $u'$  and not to  $v'$ .

Clearly,  $N(u) = \{u', x, v\}$  and by symmetry, we have, by the previous case, that  $N(u') = \{u, y, v'\}$ . Let  $G' = G - uu'$ . The graph  $G'$  has minimum degree at least two and is connected. If  $G'$  is claw-free then, by [Observation 11](#), we are done. Hence,  $G'$  has a claw  $\langle \{w_4, w_3, w_2, w_1\} \rangle$ . Clearly,  $w_2 = u$  and  $w_3 = u'$ . Hence,  $w_1 \in \{x, y, v, v'\}$ . But then  $w_1$  cannot be adjacent to both  $w_2$  and  $w_3$ , a contradiction.  $\square$

**Claim 5.** Every vertex in  $N(x) \cup N(y) - \{v_1\}$  has exactly one neighbor in  $V(G) - N[x] - N[y]$ .

**Proof.** By Claim 4 we have that every vertex in  $N(x) \cup N(y) - \{v_1\}$  has at least one neighbor in  $V(G) - N[x] - N[y]$ . Suppose that  $u$  has at least two neighbors in  $V(G) - N[x] - N[y]$ . Define  $G'_1 = \langle \{v, x, y, v_1\} \rangle$  and  $G'_2 = G - G'_1$ . Note that the graph  $G'_2$  has no degree one vertex in  $N(y) - \{v_1\}$ . As  $u$  has at least two neighbors in  $V(G) - N[x] - N[y]$ , we have that  $u$  does not have degree one in  $G'_2$  and so, by Claim 2, we have that  $v$  is adjacent to two adjacent degree two vertices  $v''$  and  $v'''$  in  $V(G) - N[x] - N[y]$ . Since  $v$  has two neighbors in  $V(G) - N[x] - N[y]$ , we have, by the same argument, that  $u$  is adjacent to two degree two vertices  $u''$  and  $u'''$  in  $V(G) - N[x] - N[y]$ . Furthermore, if  $u$  ( $v$  respectively) is adjacent to a vertex  $w$  in  $V(G) - N[x] - \{y, v'', v''', u'', u'''\}$  then  $\{u'', w, x, u\}$  ( $\{v'', w, x, v\}$  respectively) induces a claw, a contradiction. Hence,  $N(u) = \{u'', u''', v, x\}$  and  $N(v) = \{v'', v''', u, x\}$ .

We define  $G''_1 = \langle \{v'', v''', u'', u''', u, v, x, v_1\} \rangle$  and  $G''_2 = G - G''_1$ . The graph  $G''_2$  is connected, claw-free and has minimum degree at least two. Furthermore,  $G''_2 \notin \mathcal{K}$ . Hence,  $G''_2$  has a TRDS  $S$  of cardinality at most  $\frac{4(n-8)}{7}$ . The set  $\{v_1, x, v, u\} \cup S$  is a TRDS  $G$  and so  $\gamma_{tr}(G) \leq |S| + 4 \leq \frac{4n}{7} - \frac{32}{7} + \frac{28}{7} \leq \frac{4n}{7}$ , which is a contradiction.  $\square$

**Claim 6.** Exactly one of the following holds:

1. There are two distinct degree two vertices  $w$  and  $w'$  in  $V(G) - N[x] - N[y]$ , such that  $w$  is adjacent to  $u$  and  $w'$  is adjacent to  $v$ . Furthermore,  $w$  and  $w'$  have no common neighbors.
2. There is exactly one degree two vertex  $w$  in  $V(G) - N[x] - N[y]$ , such that  $w$  is adjacent to exactly one vertex in  $\{u, v\}$ .

**Proof.** Define  $G'_1 = \langle \{v_1, u, v, x\} \rangle$  and  $G'_2 = G - G'_1$ .

Case 1.  $\delta(G'_2) = 0$ .

Suppose that  $w$  is an isolated vertex in  $G'_2$ . Then  $w$  has degree two and  $N(w) = \{u, v\}$ . The path  $P' : u, w, v$  is a 2-path of length two such that  $x$  is adjacent to both  $u$  and  $v$ . This contradicts [Lemma 17](#).

Case 2.  $\delta(G'_2) = 1$ .

If a vertex of  $G'_2$  has degree one, then it must have degree at most three in  $G$ . Suppose that  $w$  is a degree one vertex of  $G'_2$ . Clearly,  $w \in V(G) - N[x] - N[y]$ . Assume first that  $w$  has degree three in  $G$ . Hence,  $w$  is adjacent to a vertex  $w' \in V(G) - N[x] - \{y, w\}$ . Thus,  $N(w) = \{w', u, v\}$ . Define  $G' = G - xv$ . The graph  $G'$  has minimum degree at least two and is connected. If  $G'$  is claw-free then, by [Observation 11](#), we are done. Hence,  $G'$  has a claw  $\langle \{w_4, w_3, w_2, w_1\} \rangle$  where  $w_2 = x$  and  $w_3 = v$ . Since  $N(x) = \{v_1, u, v\}$  we have that  $w_1 = u$ . If  $w_4 = w$  then  $w_3$  and  $w_4$  are adjacent which is

a contradiction. By Claim 5 we have that  $w_4 \in N(y) - \{v_1\}$  and so the graph  $G - \langle\{v, y, x, v_1\}\rangle$  has minimum degree at least two. This contradicts Lemma 14. We may conclude that  $\deg(w) = 2$  and that  $w$  is adjacent to exactly one vertex in the set  $\{u, v\}$ , say  $u$ . If no degree two vertex in  $V(G) - N[x] - N[y] - \{w\}$  is adjacent to  $v$  then we are done. We may assume that there is a degree two vertex  $w'$  in  $V(G) - N[x] - N[y] - \{w\}$ , such that  $w'$  is adjacent to  $v$ .

By Claim 5,  $u$  ( $v$  respectively) is not adjacent to  $w'$  ( $w$  respectively). Let  $t \in N(w) - \{u\}$  ( $t' \in N(w') - \{v\}$  respectively). The fact that  $G$  has no 2-paths of length greater than two implies that  $\deg(t) \geq 3$  and  $\deg(t') \geq 3$ . If  $t = t'$  then since  $N(w) = \{u, t\}$  and  $N(w') = \{v, t'\}$  we have that  $G$  must contain a claw, which is a contradiction. Hence,  $w$  and  $w'$  have no common neighbors.

Case 3.  $\delta(G_2) \geq 2$ .

If  $G_2'$  has no components in  $\mathcal{K}$  then, by Observation 7, we are done. Hence,  $G_2'$  has a component  $\mathcal{U}$  in  $\mathcal{K}$ . Clearly,  $N[\mathcal{U}] \subseteq V(\mathcal{U}) \cup \{v, u, v_1\}$ . In addition, also note that if  $N[y] \cap V(\mathcal{U}) \neq \emptyset$  and  $y \notin V(\mathcal{U})$  then there will be a vertex of  $V(\mathcal{U})$  that is adjacent to  $y$  and this will contradict the fact that  $N[\mathcal{U}] \subseteq V(\mathcal{U}) \cup \{v, u, v_1\}$ . Clearly,  $y \in V(\mathcal{U})$  if  $N[y] \cap V(\mathcal{U}) \neq \emptyset$ .

Suppose that  $\mathcal{U} \in \mathcal{K} - \{\mathcal{B}, C_3\}$ . If  $N[y] \cap V(\mathcal{U}) \neq \emptyset$  then, without loss of generality, we have that say  $u_1 = y$ . By Observation 8 we have that  $\deg(v_1) \geq 3$  which is a contradiction. Hence,  $V(\mathcal{U}) \subseteq V(G) - N[x] - N[y]$ . Furthermore,  $u_1$  is adjacent to say  $u$ . By Observation 8,  $u$  is adjacent to two consecutive vertices of  $\mathcal{U}$ . This contradicts Claim 5. Hence,  $\mathcal{U} \in \{\mathcal{B}, C_3\}$ .

Suppose first that  $\mathcal{U} = C_3$ . If  $N[y] \cap V(\mathcal{U}) \neq \emptyset$  then, without loss of generality, we have that  $u_1 = y$ . Hence,  $u_2$  is say  $u'$  and  $u_3$  is  $v'$ . But then neither  $u'$  nor  $v'$  is adjacent to a vertex in  $V(G) - N[x] - N[y]$ , contradicting Claim 4. It follows that  $V(\mathcal{U}) \subseteq V(G) - N[x] - N[y]$ . Clearly,  $u_1$  is adjacent to say  $u$ . If  $u$  is adjacent to a vertex  $w \in N(y)$ , then  $\langle\{w, x, u_1, u\}\rangle$  is a claw which is a contradiction. By Claim 5 we have that  $N(u) = \{v, u_1, x\}$ . If  $v$  is adjacent to no vertex of  $V(\mathcal{U})$  then  $v$  has a neighbor in  $V(G) - V(\mathcal{U}) - N[x] - N[y]$ . The graph  $G - \langle V(\mathcal{U}) \cup \{u\} \rangle$  has minimum degree at least two and has no components in  $\mathcal{K}$ . By Observation 7, we are done. Hence,  $v$  has exactly one neighbor in  $V(\mathcal{U})$ . By using the same argument that was used for  $u$  we can deduce that  $N(v) = \{x, u, z\}$ , where  $z \in V(\mathcal{U})$ . If  $v$  is adjacent to say  $u_2$ , then  $P' : u_1, u_3, u_2$  is a 2-path where  $u_1 u_2 \in E(G)$  and this contradicts our earlier assumption. Hence,  $v$  is adjacent to  $u_1$  and so  $N(v) = \{x, u, u_1\}$ . Let  $G' = G - xv$ . The graph  $G'$  has minimum degree at least two, is connected and claw-free. By Observation 11, we are done.

We may assume that  $\mathcal{U} = \mathcal{B}$ . Suppose first that  $V(\mathcal{U}) \subseteq V(G) - N[x] - N[y]$ . By Observation 8,  $u$  is adjacent to say  $u_1$ . If  $\deg(u_2) = 2$  then the path  $P' : u_1, u_2, u_3$  is a 2-path of length two such that the end vertices are adjacent, a contradiction. Hence,  $\deg(u_2) \geq 3$ . Furthermore, if  $u$  is adjacent to  $u_2$  then Claim 5 will be contradicted. Hence,  $v$  is adjacent to  $u_2$ . If  $u$  is adjacent to a vertex  $w \in N(y)$  then  $\langle\{w, x, u_1, u\}\rangle$  is a claw which is a contradiction. Hence,  $N(u) = \{u_1, v, x\}$ . By the same argument  $N(v) = \{u_2, u, x\}$ . Let  $G' = G - vu_2$ . The graph  $G'$  has minimum degree at least two, is connected and claw-free. By Observation 11, we are done. Hence,  $N[y] \cap V(\mathcal{U}) \neq \emptyset$ .

If  $u_3 = y$  then the second part of Observation 8 implies that  $v_1$  is adjacent to say  $u_1$  which is a contradiction, as  $\deg(v_1) = 2$ . Hence, without loss of generality, we have that  $u_1 = y$ . Hence,  $u_2 = v'$  and  $u_3 = u'$ . But then  $u_3$  has two neighbors,  $u_4$  and  $u_5$ , in  $V(G) - N[x] - N[y]$  and this contradicts Claim 5.  $\square$

**Claim 7.** The set  $V(G) - N[x] - N[y]$  cannot have two distinct degree two vertices  $w$  and  $w'$ , such that  $w$  is adjacent  $u$  and  $w'$  adjacent to  $v$ .

**Proof.** Suppose, to the contrary, that  $w$  and  $w'$  are two distinct degree two vertices in  $V(G) - N[x] - N[y]$ , such that  $w$  is adjacent to  $u$  and  $w'$  is adjacent to  $v$ . Let  $G'_1 = \langle\{x, v_1, u, v, w, w'\}\rangle$  and  $G'_2 = G - G'_1$ . By the first part of Claim 6 we have that  $\delta(G'_2) \geq 2$ .

Suppose first that  $G'_2$  has no component in  $\mathcal{K}$ . Then  $G'_2$  has a TRDS  $S$  of cardinality at most  $\frac{4(n-6)}{7}$ . Constructing a TRDS of  $G$  of cardinality  $|S| + 3$  will suffice since then  $\gamma_{tr}(G) \leq |S| + 3 \leq \frac{4n}{7} - \frac{24}{7} + \frac{21}{7} \leq \frac{4n}{7}$ , which is a contradiction. Consider, arbitrarily, vertices  $v_1$  and  $w$  from the set  $\{v_1, w, w'\}$ . If  $v_1$  and  $w$  have a neighbor in  $S$  then  $\{w', v\} \cup S$  is a TRDS of  $G$ . If  $v_1$  and  $w$  both have a neighbor in  $V(G'_2) - S$  and  $w'$  is adjacent to a vertex in  $S$ , then  $\{u, x\} \cup S$  is a TRDS of  $G$ . Hence, every vertex from the set  $\{v_1, w, w'\}$  has a neighbor in  $V(G'_2) - S$ . The set  $\{u, v, x\} \cup S$  is a TRDS of  $G$ . Hence,  $G'_2$  has a component  $\mathcal{U}$  in  $\mathcal{K}$ . If  $\mathcal{U} \in \mathcal{K} - \{\mathcal{B}, C_3\}$ , then, without loss of generality,  $u_1$  is adjacent to say  $w$  and, by the first part of Observation 8, we get that  $\deg(w) \geq 3$  which is a contradiction. Hence,  $\mathcal{U} \in \{\mathcal{B}, C_3\}$ . Note that  $N[\mathcal{U}] \subseteq V(\mathcal{U}) \cup \{v, u, v_1, w, w'\}$  and if  $N[y] \cap V(\mathcal{U}) \neq \emptyset$  then  $y \in V(\mathcal{U})$ .

Suppose that  $\mathcal{U} = C_3$ . Suppose first that  $V(\mathcal{U}) \subseteq V(G) - N[x] - N[y] - \{w, w'\}$ . Without loss of generality,  $u_1$  is adjacent to say  $w$ . If  $\deg(u_2) = \deg(u_3) = 2$ , then we may form  $G''_1 = \langle V(\mathcal{U}) \cup \{w\} \rangle$  and  $G''_2 = G - G''_1$ . The graph  $G''_2$  has degree at least two, is connected and  $G''_2 \notin \mathcal{K}$ . By Observation 7 we are done. Hence,  $u_2$  is adjacent to say  $w'$ . It follows that  $\deg(u_3) = 2$ . The path  $P' : u_1, u_3, u_2$  is a 2-path with adjacent end vertices which is a contradiction. Hence,  $N[y] \cap V(\mathcal{U}) \neq \emptyset$ . Suppose that  $u_1 = y$ . Hence,  $u_2 = v'$  and  $u_3 = u'$ . By Claim 4,  $v'$  ( $u'$  respectively) must be adjacent to  $w$  ( $w'$  respectively). Hence,  $n(G) = 9$ . The set  $\{x, u, v, w, v'\}$  is a TRDS of  $G$  and so  $\gamma_{tr}(G) \leq 5 < \frac{4.9}{7} = \frac{4n}{7}$ , which is a contradiction.

Hence,  $\mathcal{U} = \mathcal{B}$ . Suppose first that  $N[y] \cap V(\mathcal{U}) \neq \emptyset$ . Hence, without loss of generality,  $u_1 = y$ . Furthermore,  $u_2 = u'$  and  $u_3 = v'$ . But then  $u_3$  will be adjacent to two vertices,  $u_5$  and  $u_4$ , in  $V(G) - N[x] - N[y]$ . This contradicts Claim 5. Hence,  $V(\mathcal{U}) \subseteq V(G) - N[x] - N[y] - \{w, w'\}$ . Note that if either  $w$  or  $w'$ , say  $w$ , is adjacent to  $u_3$  then, by the second part of Observation 8, we get that  $\deg(w) \geq 3$  which is a contradiction. Hence,  $u_1$  is adjacent to say  $w$ . If  $\deg(u_2) = 2$  then  $G$  has a 2-path of length two with adjacent end vertices, a contradiction. Hence, without loss of generality,  $w'$  is adjacent to  $u_2$ . Let  $G'_1 = \langle V(\mathcal{U}) \cup \{w, w'\} \rangle$  and  $G'_2 = G - G'_1$ . The graph  $G'_2$  has degree at least two, is connected and  $G'_2 \notin \mathcal{K}$ . Hence,  $G'_2$  has a



TRDS  $S$  of cardinality at most  $\frac{4(n-7)}{7}$ . If, without loss of generality,  $u \in S$  then  $\{u_3, u_2, w'\} \cup S$  is a TRDS of  $G$ . If  $u, v \notin S$  then  $\{u_3, u_2, u_1\} \cup S$  is a TRDS of  $G$ . Hence,  $\gamma_{tr}(G) \leq |S| + 3 \leq \frac{4n}{7} - \frac{28}{7} + \frac{21}{7} \leq \frac{4n}{7}$  which is a contradiction.  $\square$

By Claims 6 and 7 and by symmetry, we may assume that say  $u$  ( $u'$  respectively) is adjacent to a degree two vertex  $w$  ( $w'$  respectively) in  $V(G) - N[x] - N[y]$ . Note that if  $w$  and  $w'$  are adjacent, then  $G$  will have a 2-path of length three which is a contradiction. Furthermore,  $v$  ( $v'$  respectively) has exactly one neighbor in  $V(G) - N[x] - N[y]$  and this neighbor has degree at least three. If  $v$  is adjacent to  $v'$  then the graph  $G - \langle \{v_1, x, u, w\} \rangle$  will have minimum degree at least two. This will contradict Lemma 13. Hence,  $v'$  is not adjacent to  $v$ . Let  $G'_1 = \langle N[y] \cup N[x] \rangle$ . We form the graph  $G'_2 = G - G'_1$ .

We first claim that no components of  $G'_2$  are in  $\mathcal{K}$ . Suppose, to the contrary, that  $\mathcal{U}$  is a component of  $G'_2$  that is in  $\mathcal{K}$ . Note that either  $v$  or  $v'$ , say  $v$ , is adjacent to a vertex of  $\mathcal{U}$ . If  $\mathcal{U} \in \mathcal{K} - \{\mathcal{B}, C_3\}$  then, by the first part of Observation 8,  $v$  is adjacent to two vertices of  $\mathcal{U}$ . This contradicts Claim 5. Hence,  $\mathcal{U} \in \{\mathcal{B}, C_3\}$ . Let  $G''_1 = \langle \{v\} \cup V(\mathcal{U}) \rangle$  and  $G''_2 = G - G''_1$ . Note that  $\delta(G''_2) \geq 2$  and  $G''_2$  has no components in  $\mathcal{K}$ . If  $\mathcal{U} = C_3$  ( $\mathcal{U} = \mathcal{B}$  respectively) then we contradict Observation 7 (second part of Observation 10 respectively). Hence,  $G'_2$  has no components in  $\mathcal{K}$ .

We also claim that  $w$  and  $w'$  are the only possible vertices of degree at most one in  $G'_2$ . Suppose that there is a vertex  $w''$  in  $V(G'_2) - \{w, w'\}$  that has degree at most one. Note that  $\deg(w'') = 3$  and  $w''$  must be adjacent to both  $v'$  and  $v$ . Since  $v$  and  $v'$  are not adjacent and by Claim 5, we have that  $G$  has a claw which is a contradiction.

Case 1.  $w = w'$ .

Let  $G'_1 = \langle N[y] \cup N[x] \cup \{w\} \rangle$  and  $G'_2 = G - G'_1$ . Clearly, the fact that  $G'_2$  has no components in  $\mathcal{K}$  implies that  $G'_2$  has no components in  $\mathcal{K}$ . Since  $w$  and  $w'$  are the only possible vertices of degree at most one in  $G'_2$ , we have that  $\delta(G'_2) \geq 2$ . Hence,  $G'_2$  has a TRDS  $S$  of cardinality at most  $\frac{4(n-8)}{7}$ . The set  $\{u, v, v', y\} \cup S$  is a TRDS of  $G$ . Hence,  $\gamma_{tr}(G) \leq |S| + 4 \leq \frac{4n}{7} - \frac{32}{7} + \frac{28}{7} \leq \frac{4n}{7}$  which is a contradiction.

Case 2.  $w \neq w'$ .

Form  $G'_2$ , from  $G'_2$ , by joining  $w$  and  $w'$ . Note that the component of  $G'_2$  that contains the edge  $ww'$  has a 2-path of length three. Hence, this component cannot be in  $\mathcal{K}$ . Furthermore, the fact that  $G'_2$  has no components in  $\mathcal{K}$  implies that  $G'_2$  has no components in  $\mathcal{K}$ . In addition,  $\delta(G'_2) \geq 2$ . Hence,  $G'_2$  has a TRDS  $S$  of cardinality at most  $\frac{4(n-7)}{7}$ . If  $w, w' \in S$  ( $w, w' \notin S$  respectively) then  $\{v', y, u', u\} \cup S$  ( $\{v', y, v_1, x\} \cup S$  respectively) is a TRDS of  $G$ . If, without loss of generality,  $w' \in S$  and  $w \notin S$  then  $\{x, u, v, u'\} \cup S$  is a TRDS of  $G$ . Hence,  $\gamma_{tr}(G) \leq |S| + 4 \leq \frac{4n}{7}$  which is a contradiction.  $\square$

**Proof of Lemma 19.** Suppose, to the contrary, that  $\delta(G) \geq 3$ . Let  $u$  be a vertex of  $G$  of minimum degree. Let  $G_1 = \langle N[u] \rangle$  and  $G_2 = G - G_1$ .

**Claim 1.**  $\langle N(u) \rangle$  contains no isolated vertices and  $G$  has no bridges.

**Proof.** Let  $x$  be an isolate in  $\langle N(u) \rangle$ . Let  $G' = G - xu$ . The graph  $G'$  has minimum degree at least two and has no components isomorphic to  $C_3$ . If  $G'$  has a claw then the center of this claw must be in  $N(u) - \{x\}$ . This contradicts the fact that  $x$  is an isolate in  $\langle N(u) \rangle$ . By Observation 11, we are done. If  $G$  has a bridge, then we may form  $G'$  by removing, from  $G$ , this bridge. The graph  $G'$  is claw-free,  $\delta(G') \geq 2$  and  $G'$  has no components in  $\mathcal{K}$ . By Observation 11, we are done.  $\square$

**Claim 2.** We may assume that  $G_2$  has no components in  $\mathcal{K}$ .

**Proof.** Suppose, to the contrary, that  $G_2$  has a component  $\mathcal{U}$  in  $\mathcal{K}$ . Note that each degree two vertex of  $\mathcal{U}$  has  $\delta - 2$  neighbors in  $N(u)$ .

Suppose first that  $\mathcal{U} \in \mathcal{K} - \{\mathcal{B}, C_3\}$ . Assume first that either  $\mathcal{U} \in \mathcal{K} - \{\mathcal{B}, C_3, C_5\}$  or  $\mathcal{U} = C_5$  and, without loss of generality,  $u_5$  has at least two neighbors in  $N(u)$ . Let  $G'_1 = \langle \{u_1, u_2, u_3, u_4\} \rangle$  and  $G'_2 = G - G'_1$ . The graph  $G'_2$  has minimum degree at least two, is connected, and  $G'_2 \notin \mathcal{K}$ . By Observation 7, we are done. Hence,  $\mathcal{U} = C_5$  and every vertex of  $\mathcal{U}$  has exactly one neighbor in  $N(u)$ . Furthermore,  $\delta = 3$ . Let  $x, y, z \in N(u)$ . Suppose, without loss of generality, that  $u_1$  is adjacent to say  $x$ . By Observation 8,  $x$  is adjacent to say  $u_2$ . If  $u_3$  is adjacent to  $x$  then  $\langle \{u, u_3, u_1, x\} \rangle$  is a claw which is a contradiction. Hence, suppose that  $u_3$  is adjacent to  $y$ . By Observation 8,  $y$  is adjacent to say  $u_4$ . If  $u_5$  is adjacent to  $x$  ( $y$  respectively) then  $\langle \{u, u_5, u_2, x\} \rangle$  ( $\langle \{u, u_5, u_3, y\} \rangle$  respectively) is a claw which is a contradiction. By the pigeonhole principle,  $u_5$  is adjacent to  $z$  and so  $z$  is adjacent to either  $u_1$  or  $u_4$  which is a contradiction. Hence,  $\mathcal{U} \in \{\mathcal{B}, C_3\}$ . Let  $x$  be a vertex of  $N(u)$  that is adjacent to say  $u_1$ .

Suppose that  $\mathcal{U} = \mathcal{B}$ . Let  $G'_1 = \langle V(\mathcal{U}) \rangle$  and  $G'_2 = G - G'_1$ . The graph  $G'_2$  is connected and has minimum degree at least two. Note that if  $G'_2 \in \mathcal{K}$  then  $G'_2 = \mathcal{B}$ . Hence, if  $G'_2 \in \mathcal{K}$  ( $G'_2 \notin \mathcal{K}$  respectively) then  $G'_2$  has a TRDS  $S$  of cardinality at most  $\frac{4(n-5)}{7} + \frac{1}{7}$ . If, without loss of generality,  $u_1$  has a neighbor in  $V(G'_2) - S$  then  $\{u_2, u_3\} \cup S$  is a TRDS of  $G$ . If  $u_1$  and  $u_2$  have a neighbor in  $S$  then  $\{u_4, u_5\} \cup S$  is a TRDS of  $G$ . Hence,  $\gamma_{tr}(G) \leq |S| + 2 \leq \frac{4(n-5)}{7} + \frac{1}{7} + \frac{14}{7} \leq \frac{4n}{7}$  which is a contradiction.

Hence,  $\mathcal{U} = C_3$ . Note that if  $x$  is adjacent to a vertex  $w \in V(G_2) - V(\mathcal{U})$ , then  $\langle \{u, w, u_1, x\} \rangle$  is a claw. Hence,  $N(x) \cap V(G_2) \subseteq V(\mathcal{U})$ . Let  $G'_1 = \langle \{u_1, u_2, u_3, x\} \rangle$  and  $G'_2 = G - G'_1$ . The graph  $G'_2$  is connected. Assume first that  $\delta(G'_2) \geq 2$ . By Observation 7,  $G'_2 \in \mathcal{K}$ . Let  $u'_1, \dots, u'_i$  be the vertices of the Hamiltonian path of  $G'_2$ . If  $G'_2 \in \mathcal{K} - \{\mathcal{B}, C_3\}$  then, without loss of generality, let  $u = u'_1$ . Furthermore,  $u'_i, u'_2 \in N(u) - \{x\}$  and  $V(G'_2) - \{u'_i, u'_1, u'_2\} \subset V(G_2)$ . But then, by the pigeonhole principle and the fact that  $\delta \geq 3$ , we have that every vertex of  $V(G'_2) - \{u'_i, u'_1, u'_2\}$  is adjacent to  $x$ . This contradicts the fact that  $N(x) \cap V(G_2) \subseteq V(\mathcal{U})$ . Hence,  $G'_2 \in \{\mathcal{B}, C_3\}$ .



If  $G'_2 = \mathcal{B}$  then  $n(G) = 9$  and the set  $\{u'_1, u'_2, u'_3, x, u_1\}$  is a *TRDS* of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq 5 < \frac{4.9}{7} = \frac{4n}{7}$  which is a contradiction.

If  $G'_2 = C_3$  then  $n(G) = 7$  and the set  $\{u, x, u_1\}$  is a *TRDS* of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq 3 < \frac{4.7}{7} = \frac{4n}{7}$  which is a contradiction.

Hence,  $\delta(G'_2) \leq 1$  and so there is a vertex  $y \in N(u) - \{x\}$  such that  $N(y) \subseteq V(\mathcal{U}) \cup \{x, u\}$ . By Claim 1,  $y$  is not isolated in  $\langle N(u) \rangle$  and so  $y$  is adjacent to  $x$ . Furthermore, the pigeonhole principle implies that  $y$  has a neighbor in  $V(\mathcal{U})$ . If we re-label  $x$  as  $y$  in  $G'_2$ , we have, by using the argument of the previous paragraph, that  $x$  has degree one in  $G'_2$ . Hence, we may assume that  $N(x) \subseteq V(\mathcal{U}) \cup \{y, u\}$ . If  $\delta = 3$  then  $\langle N(u) \rangle$  has an isolate, contradicting Claim 1. Hence,  $\delta \geq 4$ . By the pigeonhole principle,  $x$  and  $y$  have a common neighbor in  $V(\mathcal{U})$ , say  $u_1$ . Let  $G'_1 = \langle \{u_1, u_2, u_3, x, y\} \rangle$  and  $G'_2 = G - G'_1$ . Since  $\delta \geq 4$  and by Claim 1, we have that  $G'_2$  has a triangle. Furthermore,  $G'_2$  is connected and  $\delta(G'_2) \geq 2$ . Suppose first that  $G'_2 \notin \mathcal{K}$ . Hence,  $G'_2$  has a *TRDS*  $S$  of cardinality at most  $\frac{4(n-5)}{7}$ . If  $u \notin S$  or  $(u \in S \text{ respectively})$  then  $\{y, u_1\} \cup S$  ( $\{u_2, u_3\} \cup S$  respectively) is a *TRDS* of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S| + 2 \leq \frac{4n}{7} - \frac{20}{7} + \frac{14}{7} \leq \frac{4n}{7}$  which is a contradiction. Hence,  $G'_2 \in \{\mathcal{B}, C_3\}$ .

Let  $u'_1, \dots, u'_i$  be the vertices of the Hamiltonian path of  $G'_2$ . If  $G'_2 = C_3$  then set  $u = u'_1$ . If  $G'_2 = \mathcal{B}$  then let  $u \in \{u'_1, u'_3\}$ . If  $G'_2 = \mathcal{B}$  ( $G'_2 = C_3$  respectively) then  $n(G) = 10$  ( $n(G) = 8$  respectively) and so  $\{u'_1, u'_2, u'_3, u_2, u_3\}$  ( $\{u'_1, x, u_1, y\}$  respectively) is a *TRDS* of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq \frac{n}{2} < \frac{4n}{7}$  which is a contradiction.  $\square$

Suppose that  $\delta(G_2) \geq 2$  or  $V(G_2) = \emptyset$ . By Claim 2 we may assume that  $G_2$  has no components in  $\mathcal{K}$ . It follows that  $G_2$  has a *TRDS*  $S$  of cardinality at most  $\frac{4(n-\delta-1)}{7}$ . Let  $x$  be an arbitrary vertex of  $N(u)$ . If  $\{x, u\} \cup S$  is a *TRDS* of  $G$ , then  $\gamma_{\text{tr}}(G) \leq |S| + 2 \leq \frac{4n}{7} - \frac{4(\delta+1)}{7} + \frac{14}{7} \leq \frac{4n}{7}$  which is a contradiction. Hence, there is a vertex  $y \in N(u) - \{x\}$  such that  $\{u, x\} \subset N(y) \subseteq S \cup \{u, x\}$ . By switching the roles of  $y$  and  $x$  we have, by the same argument, that  $\{u, y\} \subset N(x) \subseteq S \cup \{u, y\}$ . Since  $x$  is arbitrary, we have that every vertex of  $N(u)$  has a neighbor in  $S$ . Hence,  $\{x\} \cup S$  is a *TRDS* of  $G$  and so  $\gamma_{\text{tr}}(G) < |S| + 2 \leq \frac{4n}{7} - \frac{4(\delta+1)}{7} + \frac{14}{7} \leq \frac{4n}{7}$  which is a contradiction. Hence,  $\delta(G_2) \leq 1$ . If  $\delta(G_2) = 0$  then there is a vertex  $x \in V(G_2)$  such that  $N(x) = N(u)$ . Furthermore, if  $y \in N(u)$  is adjacent to a vertex  $w \in V(G_2) - \{x\}$  then  $\langle \{w, x, u, y\} \rangle$  induces a claw. Thus,  $V(G_2) - \{x\} = \emptyset$ . Hence,  $n(G) = \delta + 2$  and the set  $\{u, y\}$  is a *TRDS* of  $G$ . Thus,  $\gamma_{\text{tr}}(G) = 2 < \frac{4(\delta+2)}{7} = \frac{4n}{7}$  which is a contradiction. Thus,  $\delta(G_2) = 1$ .

Let  $\deg_{G_2}(x) = 1$  and let  $y \in N(u)$ , where  $x$  is adjacent to every vertex of  $N(u) - \{y\}$ . Note that  $x$  is adjacent to exactly one vertex in  $V(G_2) - \{x\}$ , say  $x'$ . Furthermore, we claim if  $z'$  is an arbitrary vertex in  $N(u) - \{y\}$ , then  $z'$  is adjacent to no vertex in  $V(G_2) - \{x, x'\}$ . Suppose, to the contrary, that there is a vertex  $w \in V(G_2) - \{x, x'\}$  that is adjacent to  $z'$ . The graph  $\langle \{x, u, w, z'\} \rangle$  is a claw. Hence,  $N[z'] \subseteq \{x', x\} \cup N[u]$  for every  $z' \in N(u) - \{y\}$ .

Let  $G'_1 = \langle N[u] \cup \{x\} \rangle$  and  $G'_2 = G - G'_1$ . By Claim 1 we have that  $y$  is adjacent to a vertex in  $N(u) - \{y\}$ , say  $z$ .

Suppose first that  $\delta(G'_2) \geq 2$ . If  $G'_2$  has no components in  $\mathcal{K}$  then  $G'_2$  has a *TRDS*  $S$  of cardinality at most  $\frac{4(n-\delta-2)}{7}$ . The set  $\{z, x\} \cup S$  is a *TRDS* of  $G$  and so  $\gamma_{\text{tr}}(G) \leq |S| + 2 \leq \frac{4n}{7} - \frac{4(\delta+2)}{7} + \frac{14}{7} \leq \frac{4n}{7}$  which is a contradiction. Hence,  $G'_2$  has a component  $\mathcal{U}$  in  $\mathcal{K}$ . Now if  $x$  is adjacent to no vertex of  $V(\mathcal{U})$  then  $\mathcal{U}$  will be a component of  $G_2$ , contradicting Claim 2. We have, by the second part of [Observation 8](#), that  $x' = u_1$ . It follows that the remaining degree two vertices of  $\mathcal{U}$  must be adjacent to  $y$ .

If  $\mathcal{U} = \mathcal{B}$  ( $\mathcal{U} \in \mathcal{K} - \{\mathcal{B}, C_3\}$  respectively) then  $\langle \{u, u_4, u_2, y\} \rangle$  induces a claw. Hence,  $\mathcal{U} = C_3$  and if  $y$  is adjacent to a vertex  $w \in V(G'_2) - V(\mathcal{U})$  then  $\langle \{u, u_2, w, y\} \rangle$  induces a claw. Hence,  $n(G) = \delta + 5$ . The set  $\{z, x, y\}$  is a *TRDS* of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq 3 < \frac{4(\delta+5)}{7} = \frac{4n}{7}$  which is a contradiction.

Hence,  $\delta(G'_2) \leq 1$ . Let  $w$  be a vertex of  $G'_2$  that has degree at most one. If  $w$  is not adjacent to  $x$  then  $w \neq x'$  and, by the pigeonhole principle,  $w$  is adjacent to a vertex in  $N(u) - \{y\}$ . This contradicts the fact that no vertex in  $N(u) - \{y\}$  is adjacent to a vertex in  $V(G_2) - \{x, x'\}$ . Hence,  $x' = w$ .

Case 1.  $\deg_{G'_2}(x') = 0$ .

Hence,  $N(x') \subseteq N(u) \cup \{x\}$ . Let  $z'$  be a neighbor of  $x'$  in  $N(u) - \{y\}$ . The only vertex in  $\langle N[u] \cup \{x\} \rangle$  that can be adjacent to a vertex of  $V(G_2) - \{x, x'\}$  is  $y$ . Let  $G''_1 = \langle N[u] \cup \{x, x'\} \rangle$  and  $G''_2 = G - G''_1$ . Note that  $\delta(G''_2) \geq 2$  or  $V(G''_2) = \emptyset$ . If  $G''_2$  has a component  $\mathcal{U}$  in  $\mathcal{K}$ , then  $\mathcal{U}$  will also be a component of  $G_2$ . This contradicts Claim 2. Hence,  $G''_2$  has no components in  $\mathcal{K}$ . It follows that  $G''_2$  has a *TRDS*  $S$  of cardinality at most  $\frac{4(n-\delta-3)}{7}$ . The set  $\{u, y, z'\} \cup S$  is a *TRDS* of  $G$ . Thus,  $\gamma_{\text{tr}}(G) \leq |S| + 3 \leq \frac{4n}{7} - \frac{4(\delta+3)}{7} + \frac{21}{7} \leq \frac{4n}{7}$  which is a contradiction.

Case 2.  $\deg_{G'_2}(x') = 1$ .

Note that  $x'$  has exactly one neighbor in  $V(G'_2) - \{x'\}$ , say  $x''$ . Furthermore,  $N(x') \subseteq \{x, x''\} \cup N(u)$ . Suppose first that  $y$  is not adjacent to  $x'$ . Then  $N(x') \subseteq \{x, x''\} \cup N(u) - \{y\}$ . Let  $G' = G - x'x''$ . Clearly,  $\delta(G') \geq 2$  and  $G'$  has no components isomorphic to  $C_3$ . If  $G'$  has a claw, then the center of this claw must be in  $N(u) \cup \{x\} - \{y\}$ . But no vertex in  $N(u) \cup \{x\} - \{y\}$  is adjacent to a vertex of  $V(G_2) - \{x, x'\}$ . Hence,  $G'$  is claw-free and so, by [Observation 11](#), we are done. Hence,  $y$  is adjacent to  $x'$ .

If  $y$  is not adjacent to  $x''$  then if  $y$  is adjacent to a vertex  $w' \in V(G_2) - \{x, x', x''\}$  then  $\langle \{u, x', w', y\} \rangle$  induces a claw. Hence,  $N[y] \subseteq N[u] \cup \{x', x\}$ . It follows that  $x'x''$  is a bridge. Hence,  $y$  is adjacent to  $x''$ . Let  $G'_1 = \langle N[u] \cup \{x\} - \{y\} \rangle$  and  $G'_2 = G - G'_1$ . The graph  $G'_2$  is connected,  $\delta(G'_2) \geq 2$  and  $G'_2$  has no component in  $\mathcal{K}$ . Hence,  $G'_2$  has a *TRDS*  $S$  of cardinality at most  $\frac{4(n-\delta-1)}{7}$ . The set  $\{x, z\} \cup S$  is a *TRDS* of  $G$ . Hence,  $\gamma_{\text{tr}}(G) \leq |S| + 2 \leq \frac{4n}{7} - \frac{4(\delta+1)}{7} + \frac{14}{7} \leq \frac{4n}{7}$  which is a contradiction.  $\square$

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